

On discrete integrable equations with convex variational principles

Alexander I. Bobenko^{*,1}

Felix Günther^{*,2}

Abstract

The Lagrangian structure of two-dimensional integrable systems on quad-graphs is investigated. We give reality conditions, under which the action functionals are strictly convex. This gives in particular uniqueness of solutions of Dirichlet boundary value problems. In some cases, we discuss also the existence of solutions. The integrability of combinatorial data is studied. In addition, a connection between (Q3) and circle patterns is discussed.

2010 Mathematics Subject Classification: 37J35; 52C26; 70S05.

Keywords: discrete integrable systems, integrable quad-equations, Lagrangian formalism, variational principle, Dirichlet boundary value problem, circle patterns.

1 Introduction

This paper deals with some aspects of the variational (Lagrangian) structure of integrable systems on quad-graphs (planar graphs with quadrilateral faces), which serve as discretizations of integrable PDEs with a two-dimensional space-time [1, 6]. We identify integrability of such systems with their multidimensional consistency [6, 14]. This property was used in [1] to classify integrable systems on quad-graphs (ABS list). That paper also introduced a Lagrangian formulation for them. The variational structure of discrete integrable systems is a topic which receives increasing attention in the recent years [8, 11, 12, 15], after the pioneering work [13].

Quantization of discrete integrable systems on quad-graphs yields solvable lattice models. The consistency principle corresponds to the quantum Yang-Baxter equation. Classical discrete integrable systems on quad-graphs are then recovered in the quasi-classical limit. The corresponding action functional is derived as a quasi-classical limit of the partition function of the corresponding integrable quantum model (the Lagrangians being the quasi-classical limit of the Boltzmann weights). A quantization of circle patterns, with the corresponding quantization of the action functional introduced in [5], was carried out in [2]. The quantum “master solution” which serves as a quantization of the Laplace equation corresponding to the (Q4)-system was recently introduced in [3, 4]. Functionals obtained in the quasi-classical limit are related to the variational problems considered in the present paper.

We focus on discrete systems with convex variational principles. The latter imply in particular the uniqueness of solutions to Dirichlet boundary value problems and help to investigate the existence.

^{*}Institut für Mathematik, MA 8-3, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany. Supported by the DFG Research Unit “Polyhedral Surfaces”.

¹E-mail: bobenko@math.tu-berlin.de

²E-mail: fguenth@math.tu-berlin.de

Moreover, minimization of the corresponding functional is an effective tool to construct the corresponding solution numerically. The main result of this paper is to give conditions on parameters of (generalized) Q-systems from the ABS list that guarantee the convexity of the corresponding functionals.

The variational description we provide in Section 2 will be more general than the one established in [1] and used in [8]. Namely, our approach includes boundary terms and yields a non-integrable generalization, which in the case of (Q3) corresponds to general circle patterns with conical singularities. We will investigate the (generalized) action functionals in Section 3 and give sufficient reality conditions, under which the corresponding functional is strictly convex. The main theorem is summarized in Section 3.7. In Section 4, we also investigate the question whether quad-graphs allow an integrable labeling fulfilling our convexity conditions. The Dirichlet boundary value problems in the case of (Q4) with rectangular or rhombic lattices is solved in Section 5.

Finally, in Section 6 a geometric interpretation of generalized (Q3)-equations as Euclidean and hyperbolic circle patterns (with conical singularities) is presented. The convexity of the corresponding functionals plays a crucial role in the theory of circle patterns [5].

2 Variational description of Laplace equations of quad-equations

We consider finite connected bipartite quad-graphs \mathcal{G} (graphs such that all faces are quadrilaterals) embedded in oriented surfaces (mainly the plane). The set of vertices, edges and faces will be denoted by $V(\mathcal{G})$, $E(\mathcal{G})$ and $F(\mathcal{G})$. Possible outer faces do not count as faces of the graph.

Now choose a 2-coloring of \mathcal{G} , such that $V(\mathcal{G})$ is split into two disjoint subsets, which we call black and white vertices. Let \mathcal{G}_W be the white subgraph of \mathcal{G} and $V(\mathcal{G}_W)$ and $E(\mathcal{G}_W)$ the vertex and edge set of \mathcal{G}_W . The edges of \mathcal{G}_W are exactly the diagonals of the quadrilateral faces of \mathcal{G} . We will always assume that \mathcal{G}_W is connected and contains at least two vertices.

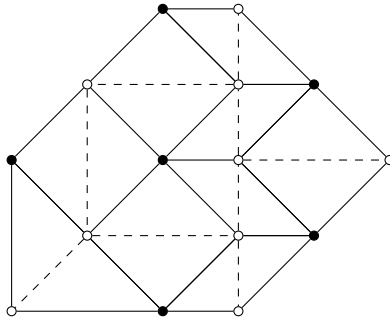


Figure 1: Bipartite quad-graph with white subgraph (dashed lines)

We will investigate labelings of $E(\mathcal{G})$ with complex numbers such that opposite edges of a quadrilateral get the same label. Usually we use the notation as shown in Figure 2. Any such labeling induces a labeling of $E(\mathcal{G}_W)$ via $\theta := \alpha - \beta$. Here α and β are the coefficients appearing in the quad-equation.

Definition. Any labeling of $E(\mathcal{G})$ with complex numbers such that opposite edges of a quadrilateral get the same label we simply call *labeling*. The labeling of $E(\mathcal{G}_W)$ described above we call *induced labeling*. If a labeling of $E(\mathcal{G}_W)$ is induced by some labeling of $E(\mathcal{G})$, we call it *integrable*.

Given a quad-graph \mathcal{G} with such a labeling of edges, we can then consider collections of equations on

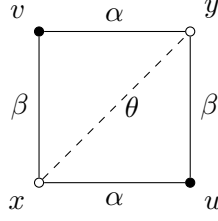


Figure 2: Quadrilateral with labeling of edges

each face of \mathcal{G} of the type

$$Q(x, u, y, v; \alpha, \beta) = 0, \quad (1)$$

where α and β are associated to the edges of the quadrilateral as in Figure 2 and x, u, y, v are complex variables assigned to the four vertices. Note that x has the meaning of a vertex as well as of the variable associated to it, but the meaning will be clear from the context. The equation Q should be one of the ABS list.

In the cases of $(Q1)_{\delta=0}$, (H1) and $(A1)_{\delta=0}$, Equation (1) possesses an additive three-leg form centered at x , i.e., it is equivalent to

$$\psi(x, u; \alpha) - \psi(x, v; \beta) = \varphi(x, y; \alpha - \beta) = \varphi(x, y; \theta),$$

for some functions ψ and φ . In the case of the other Q-equations, we have a multiplicative three-leg form, i.e., Equation (1) is equivalent to

$$\frac{\Psi(x, u; \alpha)}{\Psi(x, v; \beta)} = \Phi(x, y; \alpha, \beta) = \Phi(x, y; \theta),$$

for some functions Ψ and Φ . φ and Φ are called the long leg functions. We will be only interested in them. Since each equation from the lists A and H shares its long leg function with some equation from the list Q, we may restrict our attention to the latter. Note that $\psi = \varphi$ and $\Psi = \Phi$ for all equations from the list Q. For a list of quad-equations and three-leg functions in the case of the list Q, see the appendix.

Remark. Of course, the additive three-leg form is still true if we multiply the functions φ and ψ both with the factor i . To get real functionals later, we will not consider the long leg functions φ as they come up with the investigation of the Q-equations, but $i\varphi$ instead. For convenience, we will denote this function with φ in the following. Otherwise we would have to consider the imaginary part of the functionals.

The restriction of any solution of the system of quad-equations to the white subgraph \mathcal{G}_W then satisfies for each inner white vertex x the additive Laplace equation

$$\sum_{e=(x, y_k) \in E(\mathcal{G}_W)} \varphi(x, y_k; \theta_e) = 0 \quad (2)$$

in the case of $(Q1)_{\delta=0}$ and the multiplicative Laplace equation

$$\prod_{e=(x, y_k) \in E(\mathcal{G}_W)} \Phi(x, y_k; \theta_e) = 1 \quad (3)$$

in the case of the other Q-equations. Note that θ_e denotes the induced label on the edge e . Equation (3) corresponds to the equivalent form

$$\sum_{e=(x,y_k) \in E(\mathcal{G}_W)} \varphi(x, y_k; \theta_e) \equiv 0 \pmod{2\pi}, \quad (4)$$

where $-i\varphi = -i\varphi(x, y; \theta)$ is some branch of $\log \Phi$, i.e., a function such that $\exp(-i\varphi) = \Phi$ is satisfied. Now Equation (4) is satisfied if and only if there exists a complex number $\Theta = \Theta_x \equiv 0 \pmod{2\pi}$ such that

$$\sum_{e=(x,y_k) \in E(\mathcal{G}_W)} \varphi(x, y_k; \theta_e) + \Theta_x = 0 \quad (5)$$

holds. Especially, Θ is real. Note that Equation (2), the case of $(Q1)_{\delta=0}$, corresponds to choosing $\Theta = 0$.

Also in the case that x is not an inner white vertex of \mathcal{G} , we have a corresponding Laplace equation (5), but we do not have longer the restriction $\Theta \equiv 0 \pmod{2\pi}$ and Θ could be non-real. However, the reality conditions we will give in the sequel imply $\text{Im } \varphi \equiv 0$, so $\Theta \in \mathbb{R}$.

According to [1] and [8], for any equation from the ABS list, there exists a change of variables, $x = f(X)$, $u = f(U)$, $Y = f(Y)$, $v = f(V)$, such that in the new variables there exists a function $\Lambda = \Lambda(X, Y; \theta)$ which satisfies

$$\Lambda(X, Y; \theta) = \Lambda(Y, X; \theta), \quad (6)$$

$$\frac{\partial}{\partial X} \Lambda(X, Y; \theta) = \varphi(x, y; \theta). \quad (7)$$

Since any two branches of the natural logarithm differ only by a multiple of $2\pi i$, Λ exists for any choice $-i\varphi$ of the branch of $\log \Phi$. For convenience, we will use in the following the notation $\varphi(X, Y; \theta)$ instead of $\varphi(x, y; \theta) = \varphi(f(X), f(Y); \theta)$ and consider φ usually as a function of X, Y, θ .

The Laplace equations (5), x any vertex of \mathcal{G}_W , are the Euler-Lagrange equations for the action functional

$$S = \sum_{e=(x,y) \in E(\mathcal{G}_W)} \Lambda(X, Y; \theta_e) + \sum_{x \in V(\mathcal{G}_W)} \Theta_x X. \quad (8)$$

Thus, critical points of (8) correspond exactly to solutions of (5). Note that other choices of φ , such that $\exp(-i\varphi) = \Phi$, lead to analogous results (but Θ may change). Hence, we may always restrict to some φ of our choice.

Remark. In contrast to [1] or [8], we consider the extra terms Θ . This is due to the fact, that we consider real variables and extend φ smoothly for all of these. But the manifold of solutions of Equation (1) is not connected in real space. As a result of this, we see that φ can never be part of an additive three-leg form unless we consider $(Q1)_{\delta=0}$. In contrast, the manifold of solutions of Equation (1) is connected in projective (complex) space $(\mathbb{CP}^1)^4$ and an additive three-leg form exists.

3 Generalized action functionals of Q-equations

To define the Laplace equations (5) and to obtain the action functional (8), we do not need that the labeling of $E(\mathcal{G}_W)$ is induced by the labeling of $E(\mathcal{G})$, we could have chosen any. Then we not have to restrict Θ_x to multiples of 2π also if x is an inner white vertex. In this section we consider this more general not necessarily integrable situation. We will name the corresponding generalized

non-integrable systems also by their corresponding integrable special cases (Q1), (Q2), (Q3), (Q4). This generalized case seems to be important, for example one obtains general circle patterns (with conical singularities) in the case of (Q3). Anyway, our main interest lies in the integrable case.

In this section, we will give the generalized action functionals (8) for the individual quad equations. We always start with the three-leg form given in the appendix.

Moreover, we will give *reality conditions*, i.e., restrictions of the variables X , Y and the labels θ_e , $e = (x, y) \in E(\mathcal{G}_W)$, to certain subsets of \mathbb{C} , under which the action functional S is real strictly convex. Usually, $X, Y, \theta_e \in \mathbb{R}$. In these cases, $\Theta \in \mathbb{R}$ in general and $\Theta \in 2\pi\mathbb{Z}$ for inner white vertices in the integrable case.

Remark. For convenience, we do not make any difference in notation between convexity and concavity. In both cases we call S convex. The precise situation – whether S is convex or concave – will be clear from the respective equations.

In the case of (Q3) we will see a remarkable analogy to the Euclidean and hyperbolic circle pattern functionals of [5]. We will also obtain them as a certain limit of the action functional in the case of (Q4).

The main result of this section will be summarized in Section 3.7.

3.1 (Q1) $_{\delta=0}$

We have no change of variables, i.e., $X = x$, $Y = y$, etc. We will investigate the reality conditions $X, Y, \theta_e \in \mathbb{R}$ for all edges $e = (x, y) \in E(\mathcal{G}_W)$. The conditions $\theta_e \in \mathbb{R}\omega$ and $X, Y \in \mathbb{R}\omega + \omega'$ for any $\omega, \omega' \in \mathbb{C}, \omega \neq 0$, can be done in exactly the same way.

φ is given by

$$\varphi(X, Y; \theta) = \frac{-\theta}{X - Y},$$

which is smooth as long $X \neq Y$. The action functional can then easily be calculated:

$$S = \sum_{e=(x,y) \in E(\mathcal{G}_W)} (-\theta_e) \log |X - Y| + \sum_{x \in V(\mathcal{G}_W)} \Theta_x X. \quad (9)$$

Here log is any fixed branch of the natural logarithm. The second derivative is given by

$$D^2 S = \sum_{e=(x,y) \in E(\mathcal{G}_W)} \frac{\theta_e}{(X - Y)^2} (dX - dY)^2.$$

S is smooth and real as long $X \neq Y$ for all edges $e = (x, y)$. Otherwise S is not defined. Moreover, S is invariant under the transformation $X \mapsto X + h$ simultaneously for all vertices $x \in V(\mathcal{G}_W)$, $h \in \mathbb{R}$, if and only if

$$\sum_{x \in V(\mathcal{G}_W)} \Theta_x = 0 \quad (10)$$

is satisfied. In this case, we may restrict the variables to the subspace

$$U = \left\{ \{X\}_{x \in V(\mathcal{G}_W)} \subset \mathbb{R}^{|V(\mathcal{G}_W)|} \mid \sum_{x \in V(\mathcal{G}_W)} X = 0 \right\}. \quad (11)$$

We obtain strict convexity on the subspace U if $\theta_e \in \mathbb{R}^+$ for all $e \in E(\mathcal{G}_W)$ or $\theta_e \in \mathbb{R}^-$ for all $e \in E(\mathcal{G}_W)$. In the first case, $S \rightarrow +\infty$ if $X \rightarrow Y$ for some edge $e = (x, y)$, in the second case $S \rightarrow -\infty$.

Conversely, if S is convex, $\theta_e \geq 0$ for all $e \in E(\mathcal{G}_W)$ or $\theta_e \leq 0$ for all $e \in E(\mathcal{G}_W)$. For this, note that $\theta_e = 0$ for some edge e corresponds to deleting this edge from $E(\mathcal{G}_W)$ and

$$D^2S(X, Y) \rightarrow \frac{\theta_e}{(X - Y)^2} (dX - dY)^2$$

if $Z \rightarrow \pm\infty$ for all $z \in V(\mathcal{G}_W) \setminus \{x, y\}$, $e = (x, y)$. Here we considered $S = S(X, Y)$ as a function of only two variables, the variables corresponding to the other vertices are considered to be parameters.

3.2 (Q1) $_{\delta=1}$

We have no change of variables, i.e., $X = x$, $Y = y$, etc. We investigate the reality conditions $X, Y, \theta_e \in \mathbb{R}$ for all edges $e = (x, y) \in E(\mathcal{G}_W)$. The conditions $\theta_e \in \mathbb{R}\omega$ and $X, Y \in \mathbb{R}\omega + \omega'$ for any $\omega, \omega' \in \mathbb{C}, \omega \neq 0$, can be done in exactly the same way.

The long leg function Φ is given by

$$\Phi(X, Y; \theta) = \frac{X - Y + i\theta}{X - Y - i\theta},$$

which is smooth as long $\theta \neq 0$.

We choose

$$\varphi(X, Y; \theta) = i \log(X - Y + i\theta) - i \log(X - Y - i\theta),$$

where we now and in the remainder of this subsection consider \log to be the principle branch of the natural logarithm. φ is smooth as long $\theta \neq 0$.

$$\Lambda(X, Y; \theta) = (iX - iY - \theta) \log(X - Y + i\theta) - (iX - iY + \theta) \log(X - Y - i\theta) - 2\pi Y \operatorname{sgn}(\theta) \quad (12)$$

satisfies the Conditions (6) and (7). The action functional is then given by Formula (8). S is smooth and real if $\theta_e \neq 0$ for all edges e . The second derivative is given by

$$D^2S = \sum_{e=(x,y) \in E(\mathcal{G}_W)} \frac{2\theta_e}{(X - Y)^2 + \theta_e^2} (dX - dY)^2.$$

Thus, we can only expect convexity if we restrict to the subspace U as defined in (11). Then S is strictly convex if $\theta_e \in \mathbb{R}^+$ for all $e \in E(\mathcal{G}_W)$ or $\theta_e \in \mathbb{R}^-$ for all $e \in E(\mathcal{G}_W)$.

Conversely, if S is convex, $\theta_e \geq 0$ for all $e \in E(\mathcal{G}_W)$ or $\theta_e \leq 0$ for all $e \in E(\mathcal{G}_W)$. We can argue by

$$D^2S(X, Y) \rightarrow \frac{2\theta_e}{(X - Y)^2 + \theta_e^2} (dX - dY)^2$$

if $Z \rightarrow \pm\infty$ for all $z \in V(\mathcal{G}_W) \setminus \{x, y\}$, $e = (x, y)$. As above, we considered here $S = S(X, Y)$ as a function of only two variables, the variables corresponding to the other vertices are considered to be parameters.

To be able to restrict to the subspace U , the functional S has to be invariant under the transformation $X \mapsto X + h$ simultaneously for all vertices $x \in V(\mathcal{G}_W)$, $h \in \mathbb{R}$. This is possible if and only if

$$\sum_{x \in V(\mathcal{G}_W)} \Theta_x = 2\pi \sum_{e \in E(\mathcal{G}_W)} \operatorname{sgn}(\theta_e). \quad (13)$$

Note that

$$\begin{aligned} 0 &> \varphi(X, Y; \theta) > -2\pi \text{ if } \theta > 0, \\ 0 &< \varphi(X, Y; \theta) < +2\pi \text{ if } \theta < 0. \end{aligned}$$

So according to Equations (5),

$$\begin{aligned} \forall x \in V(\mathcal{G}_W) : 0 < \Theta_x < +2\pi \deg_{\mathcal{G}_W}(x) \text{ if } \theta_e > 0 \forall e \in E(\mathcal{G}_W), \\ \forall x \in V(\mathcal{G}_W) : 0 > \Theta_x > -2\pi \deg_{\mathcal{G}_W}(x) \text{ if } \theta_e < 0 \forall e \in E(\mathcal{G}_W) \end{aligned}$$

is necessary to obtain solutions.

3.3 (Q2)

We consider the following change of variables: $x = X^2$, $y = Y^2$, etc. As above, we will investigate the reality conditions $X, Y, \theta_e \in \mathbb{R}$ for all edges $e = (x, y) \in E(\mathcal{G}_W)$. The conditions $X, Y, \theta_e \in \mathbb{R}\omega$ for any $\omega \in \mathbb{C} \setminus \{0\}$ can be done in exactly the same way.

The long leg function Φ is given by

$$\Phi(X, Y; \theta) = \frac{(X - Y + i\theta)(X + Y + i\theta)}{(X - Y - i\theta)(X + Y - i\theta)},$$

which is smooth as long $\theta \neq 0$.

We choose

$$\varphi(X, Y; \theta) = i \log(X - Y + i\theta) - i \log(X - Y - i\theta) + i \log(X + Y + i\theta) - i \log(X + Y - i\theta),$$

where we now and in the remainder of this subsection consider \log to be the principle branch of the natural logarithm.

$$\begin{aligned} \Lambda(X, Y; \theta) &= (iX - iY - \theta) \log(X - Y + i\theta) - (iX - iY + \theta) \log(X - Y - i\theta) \\ &\quad + (iX + iY - \theta) \log(X + Y + i\theta) - (iX + iY + \theta) \log(X + Y - i\theta) - 2\pi Y \operatorname{sgn}(\theta) \end{aligned} \quad (14)$$

satisfies the Conditions (6) and (7). The action functional is then given by Formula (8). S is smooth and real if $\theta_e \neq 0$ for all edges e . The second derivative is given by

$$D^2 S = \sum_{e=(x,y) \in E(\mathcal{G}_W)} \frac{2\theta_e}{(X - Y)^2 + \theta_e^2} (dX - dY)^2 + \sum_{e=(x,y) \in E(\mathcal{G}_W)} \frac{2\theta_e}{(X + Y)^2 + \theta_e^2} (dX + dY)^2.$$

We have strict convexity on $\mathbb{R}^{|V(\mathcal{G}_W)|}$ if $\theta_e \in \mathbb{R}^+$ for all $e \in E(\mathcal{G}_W)$ or $\theta_e \in \mathbb{R}^-$ for all $e \in E(\mathcal{G}_W)$.

Conversely, if S is convex, $\theta_e \geq 0$ for all $e \in E(\mathcal{G}_W)$ or $\theta_e \leq 0$ for all $e \in E(\mathcal{G}_W)$. We simply argue by

$$D^2 S(X, Y) \rightarrow \frac{2\theta_e}{(X - Y)^2 + \theta_e^2} (dX - dY)^2 + \frac{2\theta_e}{(X + Y)^2 + \theta_e^2} (dX + dY)^2$$

if $Z \rightarrow \pm\infty$ for all $z \in V(\mathcal{G}_W) \setminus \{x, y\}$, $e = (x, y)$. We considered $S = S(X, Y)$ as a function of only two variables, the variables corresponding to the other vertices are considered to be parameters.

In the same way as in the case of (Q1) $_{\delta=1}$,

$$\begin{aligned} \forall x \in V(\mathcal{G}_W) : 0 < \Theta_x < +4\pi \deg_{\mathcal{G}_W}(x) \text{ if } \theta_e > 0 \forall e \in E(\mathcal{G}_W), \\ \forall x \in V(\mathcal{G}_W) : 0 > \Theta_x > -4\pi \deg_{\mathcal{G}_W}(x) \text{ if } \theta_e < 0 \forall e \in E(\mathcal{G}_W) \end{aligned}$$

is necessary to obtain solutions.

3.4 (Q3)_{δ=0}

We consider the following change of variables: $x = \exp(X)$, $y = \exp(Y)$, etc. Note that the quad-equation does not change if we add multiples of 2π to α or β , in this sense θ is defined only up to a multiple of 2π . In this subsection, we will always have $\theta_e \in \mathbb{R}$ for all edges e . For convenience, we fix all the labels θ in such a way that $\theta \in [0, 2\pi)$. We could fix the labels also in a different way, such that not all θ are in $[0, 2\pi)$, the same argumentation would work (but Θ may change). First, we will investigate the reality conditions $X, Y \in \mathbb{R}$ for all edges $e = (x, y) \in E(\mathcal{G}_W)$. Note that the conditions $X \in \mathbb{R} + \omega$ for all $x \in V(\mathcal{G})$, $\omega \in \mathbb{C}$ (e.g. $\omega = \pi i$), are equivalent.

The long leg function Φ is given by

$$\Phi(X, Y; \theta) = \exp(-i\theta) \frac{\sinh(\frac{X-Y+i\theta}{2})}{\sinh(\frac{X-Y-i\theta}{2})}, \quad (15)$$

which is smooth as long $\theta \not\equiv 0 \pmod{2\pi}$.

We choose

$$\varphi(X, Y; \theta) = i \log \left(-\frac{\sinh(\frac{X-Y+i\theta}{2})}{\sinh(\frac{X-Y-i\theta}{2})} \right) - (\pi - \theta), \quad (16)$$

where \log is the principle branch of the natural logarithm. φ is smooth if $\theta \not\equiv 0 \pmod{2\pi}$.

Using the notation $\theta^* := \pi - \theta$ from now on, we obtain for the action functional

$$\begin{aligned} S = & \sum_{e=(x,y) \in E(\mathcal{G}_W)} \{ \text{Im Li}_2(\exp(X - Y + i\theta_e)) + \text{Im Li}_2(\exp(Y - X + i\theta_e)) - \theta_e^*(X + Y) \} \\ & + \sum_{x \in V(\mathcal{G}_W)} \Theta_x X, \end{aligned} \quad (17)$$

which can be easily checked. Some elementary calculation yields

$$D^2 S = \sum_{e=(x,y) \in E(\mathcal{G}_W)} \frac{\sin(\theta_e)}{\cosh(X - Y) - \cos(\theta_e)} (dX - dY)^2.$$

Thus, we can only expect convexity if we restrict the variables X to the subspace U as defined in (11). Then S is strictly convex if $\theta_e \in (0, \pi)$ for all $e \in E(\mathcal{G}_W)$ or $\theta_e \in (\pi, 2\pi)$ for all $e \in E(\mathcal{G}_W)$. Since θ was a priori only defined up to a multiple of 2π , we can state more generally that S is strictly convex if $\exp(i\theta_e) \in S_+^1$ for all $e \in E(\mathcal{G}_W)$ or $\exp(i\theta_e) \in S_-^1$ for all $e \in E(\mathcal{G}_W)$. Here $S_\pm^1 := \{z \in \mathbb{C} \mid |z| = 1, \pm \text{Im}(z) > 0\}$.

Conversely, if S is convex, $\theta_e \in [0, \pi]$ for all $e \in E(\mathcal{G}_W)$ or $\theta_e \in [\pi, 2\pi]$ for all $e \in E(\mathcal{G}_W)$ (up to multiples of 2π). As above, we argue by

$$D^2 S(X, Y) \rightarrow \frac{\sin(\theta_e)}{\cosh(X - Y) - \cos(\theta_e)} (dX - dY)^2$$

if $Z \rightarrow \pm\infty$ for all $z \in V(\mathcal{G}_W) \setminus \{x, y\}$, $e = (x, y)$. Here $S = S(X, Y)$ is considered as a function of only two variables, the variables corresponding to the other vertices are considered to be parameters.

To be able to restrict to the subspace U , the functional S has to be invariant under the transformation $X \mapsto X + h$ simultaneously for all vertices $x \in V(\mathcal{G}_W)$, $h \in \mathbb{R}$. This is possible if and only if

$$\sum_{x \in V(\mathcal{G}_W)} \Theta_x = \sum_{e \in E(\mathcal{G}_W)} 2\theta_e^* \quad (18)$$

is satisfied.

Let $t \in (0, \pi)$. Then

$$\varphi(X, Y; \pi + t) = -\varphi(X, Y; \pi - t). \quad (19)$$

Hence, the Laplace equations (5) with $\theta_e = \pi - t_e$ and $\theta'_e = \pi + t_e$ are equivalent if one chooses $\Theta_x = -\Theta'_x$. Thus, we may restrict our attention to the convexity condition $\exp(i\theta_e) \in S_+^1$ for all $e \in E(\mathcal{G}_W)$.

In this case, $-2\pi < \varphi(X, Y; \theta) < 0$, so $0 < \Theta_x < 2\pi \deg_{\mathcal{G}_W}(x)$ for all vertices x is necessary to obtain solutions by Equations (5). Therefore, our action functional (17) can be identified with the Euclidean circle pattern functional in [5]. Also the scaling Condition (18) can be found there.

In the following, we will consider different reality conditions. We need that the white subgraph \mathcal{G}_W is bipartite. Now choose a partition of the vertices into two subsets V_1 and V_2 , such that each edge contains a point of each set as an endpoint. We consider the reality conditions $X \in \mathbb{R}$ and $Y \in \mathbb{R} + \pi i$, if $x \in V_1, y \in V_2$.

Let us write $X' = X$ if $x \in V_1$ and $X' = X - \pi i$ if $x \in V_2$. Moreover, $\theta' := \theta \pm \pi$ (sign is chosen such that $\theta' \in [0, 2\pi)$). Then

$$\Phi(X, Y; \theta) = \exp(-i\theta') \frac{\sinh(\frac{X' - Y' + i\theta'}{2})}{\sinh(\frac{X' - Y' - i\theta'}{2})}.$$

Thus, we can argue exactly as above with parameters θ' instead of θ . Hence, we obtain exactly the same convexity conditions. The action functional in the case of $\theta \in (\pi, 2\pi)$ is given by (17), replacing X, Y, θ by X', Y', θ' . Because of the remark to Equation (19), we can take (17) as action functional in the case of $\theta \in (0, \pi)$, if we replace X, Y by X', Y' , θ by θ^* and Θ by $-\Theta$.

3.5 (Q3) $_{\delta=1}$

We consider $x = \cosh(X)$, $y = \cosh(Y)$, etc., as change of variables. First, we will investigate the reality conditions $X, Y, \theta_e \in \mathbb{R}$ for all edges $e = (x, y) \in E(\mathcal{G}_W)$. It is easy to see that the conditions $X \in \mathbb{R} + k\pi i$ for all vertices x , $k \in \mathbb{Z}$ fixed, are equivalent.

The long leg function is given by

$$\Phi(X, Y; \theta) = \frac{\sinh(\frac{X - Y + i\theta}{2}) \sinh(\frac{X + Y + i\theta}{2})}{\sinh(\frac{X - Y - i\theta}{2}) \sinh(\frac{X + Y - i\theta}{2})}. \quad (20)$$

We choose

$$\varphi(X, Y; \theta) = i \log \left(-\frac{\sinh(\frac{X - Y + i\theta}{2})}{\sinh(\frac{X - Y - i\theta}{2})} \right) + i \log \left(-\frac{\sinh(\frac{X + Y + i\theta}{2})}{\sinh(\frac{X + Y - i\theta}{2})} \right), \quad (21)$$

where \log is the principle branch of the natural logarithm. Φ and φ are smooth if $\theta \not\equiv 0 \pmod{2\pi}$.

The same calculations as in Section 3.4 deliver

$$\begin{aligned} S = & \sum_{e=(x,y) \in E(\mathcal{G}_W)} \{ \text{Im Li}_2(\exp(X - Y + i\theta_e)) + \text{Im Li}_2(\exp(Y - X + i\theta_e)) \} \\ & + \sum_{e=(x,y) \in E(\mathcal{G}_W)} \{ \text{Im Li}_2(\exp(X + Y + i\theta_e)) + \text{Im Li}_2(\exp(-X - Y + i\theta_e)) \} + \sum_{x \in V(\mathcal{G}_W)} \Theta_x X, \end{aligned} \quad (22)$$

as the action functional and

$$D^2S = \sum_{e=(x,y) \in E(\mathcal{G}_W)} \left\{ \frac{\sin(\theta_e)}{\cosh(X-Y) - \cos(\theta_e)} (dX - dY)^2 \right\} + \sum_{e=(x,y) \in E(\mathcal{G}_W)} \left\{ \frac{\sin(\theta_e)}{\cosh(X+Y) - \cos(\theta_e)} (dX + dY)^2 \right\}. \quad (23)$$

Thus, S as a function of the variables X is strictly convex on $\mathbb{R}^{|V(\mathcal{G}_W)|}$ if $\exp(i\theta_e) \in S_+^1$ for all $e \in E(\mathcal{G}_W)$ or $\exp(i\theta_e) \in S_-^1$ for all $e \in E(\mathcal{G}_W)$.

Conversely, if S is convex, $\exp(i\theta_e) \notin S_-^1$ for all $e \in E(\mathcal{G}_W)$ or $\exp(i\theta_e) \notin S_+^1$ for all $e \in E(\mathcal{G}_W)$. For this, note that

$$D^2S(X, Y) \rightarrow \frac{\sin(\theta_e)}{\cosh(X-Y) - \cos(\theta_e)} (dX - dY)^2 + \frac{\sin(\theta_e)}{\cosh(X+Y) - \cos(\theta_e)} (dX + dY)^2$$

if $Z \rightarrow \pm\infty$ for all $z \in V(\mathcal{G}_W) \setminus \{x, y\}$, $e = (x, y)$. As above, $S = S(X, Y)$ is considered as a function of only two variables, the variables corresponding to the other vertices are considered to be parameters.

Since $|\varphi(X, Y; \theta)| < 2\pi$ if $\theta \not\equiv 0 \pmod{2\pi}$, $|\Theta_x| < 2\pi \deg_{\mathcal{G}_W}(x)$ is necessary to obtain solutions by Equations (5).

If $t \in (0, \pi)$, then

$$\varphi(X, Y; \pi + t) = -\varphi(X, Y; \pi - t), \quad (24)$$

so the Laplace equations (5) with $\theta_e = \pi - t_e$ and $\theta'_e = \pi + t_e$ are equivalent if one chooses $\Theta_x = -\Theta'_x$. Thus, we may restrict our attention to the convexity condition $\exp(i\theta_e) \in S_+^1$ for all $e \in E(\mathcal{G}_W)$.

In this case, $\varphi(X, Y; \theta) < 0$ if $X, Y < 0$, so $\Theta_x > 0$ for all vertices x is necessary to obtain solutions with $X < 0$ for all $x \in V(\mathcal{G}_W)$. Therefore our action functional (22) can be identified with the hyperbolic circle pattern functional in [5].

If \mathcal{G}_W is bipartite, we take a partition of the set of vertices into V_1 and V_2 as in Section 3.4. We consider the reality conditions $X \in \mathbb{R}$ and $Y \in \mathbb{R} + \pi i$, if $x \in V_1, y \in V_2$.

Let us write $X' = X$ if $x \in V_1$ and $X = X' - \pi i$ if $x \in V_2$. Moreover, $\theta' := \theta \pm \pi$ (sign is chosen such that $\theta' \in [0, 2\pi)$). Then

$$\Phi(X, Y; \theta) = \frac{\sinh(\frac{X'-Y'+i\theta'}{2}) \sinh(\frac{X'+Y'+i\theta'}{2})}{\sinh(\frac{X'-Y'-i\theta'}{2}) \sinh(\frac{X'+Y'-i\theta'}{2})}.$$

Arguing exactly as in Section 3.4, we obtain exactly the same convexity conditions. The action functional in the case of $\theta \in (\pi, 2\pi)$ is given by (22), replacing X, Y, θ by X', Y', θ' . Because of the remark to Equation (24), we can take (22) as action functional in the case of $\theta \in (0, \pi)$, if we replace X, Y by X', Y' , θ by θ^* and Θ by $-\Theta$.

3.6 (Q4)

Remark. This section uses several facts of complex analysis and the theory of elliptic functions, most of them one can find in [9]. But in contrast to [9], we use the Jacobi theta functions $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 = \vartheta_0$

with half-period ratio τ defined by the following, where $h^k := \exp(i\pi\tau k)$ for $k \in \mathbb{R}$:

$$\begin{aligned}\vartheta_1(v) &= 2\{h^{\frac{1}{4}} \sin(v) - h^{\frac{9}{4}} \sin(v) + h^{\frac{25}{4}} \sin(v) \mp \dots\}, \\ \vartheta_2(v) &= 2\{h^{\frac{1}{4}} \cos(v) + h^{\frac{9}{4}} \cos(v) + h^{\frac{25}{4}} \cos(v) + \dots\}, \\ \vartheta_3(v) &= 1 + 2\{h \cos(2v) + h^4 \cos(4v) + h^9 \cos(6v) + \dots\}, \\ \vartheta_4(v) &= 1 - 2\{h \cos(2v) - h^4 \cos(4v) + h^9 \cos(6v) \mp \dots\}.\end{aligned}$$

A scaling of the argument by the factor π connects the definition above with the definition in [9].

We consider the following change of variables: $x = \text{sn}(-iX + \pi/2)$, $y = \text{sn}(-iY + \pi/2)$, etc., where sn is the Jacobi elliptic function sn with modulus $\kappa = \vartheta_2^2(0)\vartheta_3^{-2}(0)$. In the following, we will only consider purely imaginary τ , corresponding to rectangular lattices.

We will investigate the reality conditions $X, Y, \theta_e \in \mathbb{R}$ for all edges $e = (x, y) \in E(\mathcal{G}_W)$. Clearly, the conditions $X \in \mathbb{R} + k\pi i$ for all vertices x , $k \in \mathbb{R}$ fixed, can also be chosen.

The long leg function Φ is given by

$$\Phi(X, Y; \theta) = \frac{\vartheta_1\left(\frac{X+Y+i\theta}{2i}\right) \vartheta_4\left(\frac{X+Y+i\theta}{2i}\right) \vartheta_1\left(\frac{X-Y+i\theta}{2i}\right) \vartheta_4\left(\frac{X-Y+i\theta}{2i}\right)}{\vartheta_1\left(\frac{X+Y-i\theta}{2i}\right) \vartheta_4\left(\frac{X+Y-i\theta}{2i}\right) \vartheta_1\left(\frac{X-Y-i\theta}{2i}\right) \vartheta_4\left(\frac{X-Y-i\theta}{2i}\right)} \quad (25)$$

which is smooth as long $\theta \not\equiv 0 \pmod{2\pi}$. Since Φ is 2π -periodic in the θ -variable, we may fix the labels $\theta \in [0, 2\pi)$.

Now fix for a moment $\theta \not\equiv 0 \pmod{2\pi}$. Let U be an open strip along the real line such that for all $u \in U$, $\text{Im}(u \pm i\theta) \not\equiv 0 \pmod{2\pi}$. Then for $j \in \{1, 4\}$, the function

$$u \mapsto (-1)^j \frac{\vartheta_j\left(\frac{u+i\theta}{2i}\right)}{\vartheta_j\left(\frac{u-i\theta}{2i}\right)}$$

is holomorphic and has no zeroes in U . Moreover, it takes the value 1 if $u = 0$. Thus, there is a holomorphic $s_j(u, \theta)$ such that

$$\exp(s_j(u, \theta)) = (-1)^j \frac{\vartheta_j\left(\frac{u+i\theta}{2i}\right)}{\vartheta_j\left(\frac{u-i\theta}{2i}\right)}$$

and $s_j(0; \theta) = 0$. Then $s_j(Z; \theta)$ considered as a function of $Z, \theta \in \mathbb{R}$ is smooth as long $\theta \not\equiv 0 \pmod{2\pi}$. We now define

$$\log \left((-1)^j \frac{\vartheta_j\left(\frac{Z+i\theta}{2i}\right)}{\vartheta_j\left(\frac{Z-i\theta}{2i}\right)} \right) := s_j(Z; \theta)$$

if $Z, \theta \in \mathbb{R}$ and $\theta \not\equiv 0 \pmod{2\pi}$.

Hence, we can choose

$$\begin{aligned}\varphi(X, Y; \theta) &= i \log \left(-\frac{\vartheta_1\left(\frac{X+Y+i\theta}{2i}\right)}{\vartheta_1\left(\frac{X+Y-i\theta}{2i}\right)} \right) + i \log \left(\frac{\vartheta_4\left(\frac{X+Y+i\theta}{2i}\right)}{\vartheta_4\left(\frac{X+Y-i\theta}{2i}\right)} \right) \\ &\quad + i \log \left(-\frac{\vartheta_1\left(\frac{X-Y+i\theta}{2i}\right)}{\vartheta_1\left(\frac{X-Y-i\theta}{2i}\right)} \right) + i \log \left(\frac{\vartheta_4\left(\frac{X-Y+i\theta}{2i}\right)}{\vartheta_4\left(\frac{X-Y-i\theta}{2i}\right)} \right),\end{aligned} \quad (26)$$

which is smooth as long $\theta \not\equiv 0 \pmod{2\pi}$.

For $j \in \{1, 4\}$, define the function $I_j : \mathbb{R} \rightarrow \mathbb{C}$ via

$$I_j(Z; \theta) = i \int_0^Z \log \left((-1)^j \frac{\vartheta_j\left(\frac{u+i\theta}{2i}\right)}{\vartheta_j\left(\frac{u-i\theta}{2i}\right)} \right) du,$$

where we integrate along the real line. I is defined and smooth in both variables if $\theta \not\equiv 0 \pmod{2\pi}$, and $I(0; \theta) = 0$. Since $(-1)^j \vartheta_j$ is an even function, it easily follows that I is even. Thus,

$$\Lambda(X, Y; \theta) = I_1(X + Y; \theta) + I_4(X + Y; \theta) + I_1(X - Y; \theta) + I_4(X - Y; \theta) \quad (27)$$

satisfies the Conditions (6) and (7). The action functional is then given by Formula (8) in the new variables X .

Let $\omega := \pi i$ and $\omega' := \tau\omega = -t$ for some $t \in \mathbb{R}^+$. We introduce the meromorphic function

$$\tilde{\zeta}(u) := \frac{\pi}{2\omega} \frac{\vartheta_1'(v)}{\vartheta_1(v)}, \quad v = \frac{\pi u}{2\omega}.$$

Note that

$$\tilde{\zeta}(u) = \zeta(u) - \frac{\eta u}{\omega},$$

where $\eta := \zeta(\omega)$ and ζ is the Weierstrass zeta-function corresponding to the half-periods ω, ω' of the Weierstrass \wp -function.

Using additionally that

$$\frac{\vartheta_4\left(\frac{X+Y+i\theta}{2i}\right) \vartheta_4\left(\frac{X-Y+i\theta}{2i}\right)}{\vartheta_4\left(\frac{X+Y-i\theta}{2i}\right) \vartheta_4\left(\frac{X-Y-i\theta}{2i}\right)} = \frac{\vartheta_1\left(\frac{X+Y+i\theta}{2i} - \frac{\pi\tau}{2}\right) \vartheta_1\left(\frac{X-Y+i\theta}{2i} + \frac{\pi\tau}{2}\right)}{\vartheta_1\left(\frac{X+Y-i\theta}{2i} - \frac{\pi\tau}{2}\right) \vartheta_1\left(\frac{X-Y-i\theta}{2i} + \frac{\pi\tau}{2}\right)},$$

we obtain

$$\begin{aligned} D^2 S &= i \sum_{e=(x,y) \in E(\mathcal{G}_W)} \left\{ \tilde{\zeta}(X + Y + i\theta_e) - \tilde{\zeta}(X + Y - i\theta_e) \right\} (dX + dY)^2 \\ &+ i \sum_{e=(x,y) \in E(\mathcal{G}_W)} \left\{ \tilde{\zeta}(X + Y + t + i\theta_e) - \tilde{\zeta}(X + Y + t - i\theta_e) \right\} (dX + dY)^2 \\ &+ i \sum_{e=(x,y) \in E(\mathcal{G}_W)} \left\{ \tilde{\zeta}(X - Y + i\theta_e) - \tilde{\zeta}(X - Y - i\theta_e) \right\} (dX - dY)^2 \\ &+ i \sum_{e=(x,y) \in E(\mathcal{G}_W)} \left\{ \tilde{\zeta}(X - Y - t + i\theta_e) - \tilde{\zeta}(X - Y - t - i\theta_e) \right\} (dX - dY)^2. \end{aligned} \quad (28)$$

Let Γ be the lattice spanned by $2\omega, 2\omega'$. Then $\Gamma = \overline{\Gamma}$ and $\tilde{\zeta}(\overline{u}) = \overline{\tilde{\zeta}(u)}$ for all $u \in \mathbb{C}$. Using $s_j(0; \theta) = 0$, it follows that $\text{Im } \varphi \equiv 0$ and $\text{Im } S \equiv 0$ (using $I_j(0; \theta) = 0$).

As above, let $h^k := \exp(i\pi\tau k)$ for any $k \in \mathbb{R}$. Moreover, $z^{\pm 2} := \exp(\pm u)$.

$$\begin{aligned} \tilde{\zeta}(u) &= \frac{\pi}{2\omega} \cot\left(\frac{\pi u}{2\omega}\right) + \frac{i\pi}{\omega} \sum_{k=1}^{\infty} \left(\frac{h^2 k z^{-2}}{1 - h^2 k z^{-2}} - \frac{h^2 k z^2}{1 - h^2 k z^2} \right) \\ &= -\frac{i}{2} \cot\left(-\frac{i u}{2}\right) + \sum_{k=1}^{\infty} \frac{\exp(2tk) \exp(-u) - \exp(2tk) \exp(u)}{\exp(4tk) - \exp(2tk)(\exp(u) + \exp(-u)) + 1} \\ &= \frac{1}{2} \coth\left(\frac{u}{2}\right) + \sum_{k=1}^{\infty} \frac{\sinh(u)}{\cosh(u) - \cosh(2tk)}. \end{aligned} \quad (29)$$

The series (29) converges absolutely if the denominator never vanishes. Especially, the series converges for $u = X \pm Y \pm i\theta$ if $\theta \not\equiv 0 \pmod{\pi}$.

Some elementary calculations yield

$$-\frac{\tilde{\zeta}(X \pm Y + i\theta) - \tilde{\zeta}(X \pm Y - i\theta)}{i \sin \theta} = \frac{1}{\cosh(X \pm Y) - \cos(\theta)} + 2 \sum_{k=1}^{\infty} \frac{\cosh(X \pm Y) \cosh(2kt) - \cos(\theta)}{\sinh^2(X \pm Y) + \cosh^2(2kt) + \cos^2(\theta) - 2 \cosh(X \pm Y) \cosh(2kt) \cos(\theta)}. \quad (30)$$

The right hand side is a positive real number if $\theta \not\equiv 0 \pmod{\pi}$. For this, note that in this case $\cos(\theta) < 1 \leq \cosh(X \pm Y) \leq \cosh(X \pm Y) \cosh(2kt)$ and

$$\begin{aligned} & \sinh^2(X \pm Y) + \cosh^2(2kt) + \cos^2(\theta) - 2 \cosh(X \pm Y) \cosh(2kt) \cos(\theta) \\ & \geq 2 \cosh(X \pm Y) \cosh(2kt) (1 - \cos(\theta)) + \cos^2(\theta) - 1 \\ & = (1 - \cos(\theta)) (2 \cosh(X \pm Y) \cosh(2kt) - \cos(\theta) - 1) \\ & > 0. \end{aligned}$$

Thus, S is strictly convex if $\theta_e \in (0, \pi)$ for all edges e or $\theta_e \in (\pi, 2\pi)$ for all edges e . Since the labels θ are a priori only defined up to multiples of 2π , we can state this equivalently as: S is strictly convex if $\exp(i\theta_e) \in S_+^1$ for all $e \in E(\mathcal{G}_W)$ or $\exp(i\theta_e) \in S_-^1$ for all $e \in E(\mathcal{G}_W)$.

Note that S can be strictly convex also if there are labels θ_e on edges e satisfying $\exp(i\theta_e) \in S_+^1$ as well as labels $\theta_{e'}$ with $\exp(i\theta_{e'}) \in S_-^1$ (even in the integrable case). For example, consider as the white subgraph two vertices x, y connected by an edge e with the label $\theta_e = -\pi/2$ and $4k+1$ neighbors of x and $4k+1$ neighbors of y only connected by an edge to x or y , respectively. Each of the other edges gets the label $\pi/2$. It is easy to construct a bipartite planar quad-graph having the given graph as the white subgraph and a labeling such that the given one is induced. The function $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$f(u) := i \left(\tilde{\zeta}(u + \frac{\pi}{2}i) - \tilde{\zeta}(u - \frac{\pi}{2}i) \right),$$

is periodic and positive. So there are positive constants K_+, K_- such that $K_+ \geq f(u) \geq K_-$ for all real u . Choosing the integer k such that $4k > K_+/K_- - 1$ yields a strictly convex action functional S .

In the case that \mathcal{G}_W is bipartite, we take a partition of the set of vertices into V_1 and V_2 as in Sections 3.4 and 3.5. We consider the reality conditions $X \in \mathbb{R}$ and $Y \in \mathbb{R} + \pi i$, if $x \in V_1, y \in V_2$.

Let us write $X' = X$ if $x \in V_1$ and $X' = X - \pi i$ if $x \in V_2$; $\theta' := \theta \pm \pi$ (sign is chosen such that $\theta' \in [0, 2\pi)$). Then

$$\Phi(X, Y; \theta) = \frac{\vartheta_1\left(\frac{X'+Y'+i\theta'}{2i}\right) \vartheta_4\left(\frac{X'+Y'+i\theta'}{2i}\right) \vartheta_1\left(\frac{X'-Y'+i\theta'}{2i}\right) \vartheta_4\left(\frac{X'-Y'+i\theta'}{2i}\right)}{\vartheta_1\left(\frac{X'+Y'-i\theta'}{2i}\right) \vartheta_4\left(\frac{X'+Y'-i\theta'}{2i}\right) \vartheta_1\left(\frac{X'-Y'-i\theta'}{2i}\right) \vartheta_4\left(\frac{X'-Y'-i\theta'}{2i}\right)}.$$

Thus, we obtain the action functional by replacing X, Y, θ by X', Y', θ' and therefore exactly the same convexity conditions as above.

Remark. If we follow [1], the long leg function Φ is given by

$$\Phi(X, Y; \theta) = \frac{\sigma(X - Y + i\theta) \sigma(X + Y + i\theta)}{\sigma(X - Y - i\theta) \sigma(X + Y - i\theta)}.$$

Note that in this formulation, $i\theta := A - B$, where A, B are certain values fulfilling $\wp(A) = \alpha, \wp(B) = \beta$.

Let $\omega = \pi i$, $\omega' = -t$ be the half-periods of \wp as above and let $\eta := \zeta(\omega)$, $\eta' := \zeta(\omega')$. In the classification of integrable quad-equations in [1], (Q4) corresponds to the polynomial $r(x) = 4x^3 - g_2x - g_3$.

$$x = \wp(X), r(x) = \wp'(X)^2$$

fulfills this equation. Applying the transformation $x \mapsto x - \eta/\omega$, we obtain an equivalent quad-equation. This leads to the following long leg function:

$$\Phi(X, Y; \theta) = \frac{\vartheta_1\left(\frac{X+Y+i\theta}{2i}\right) \vartheta_1\left(\frac{X-Y+i\theta}{2i}\right)}{\vartheta_1\left(\frac{X+Y-i\theta}{2i}\right) \vartheta_1\left(\frac{X-Y-i\theta}{2i}\right)}.$$

Here $i\theta := A - B$, where A, B are certain values fulfilling $\wp(A) = \alpha - \frac{\eta}{\omega}$, $\wp(B) = \beta - \frac{\eta}{\omega}$. For the second derivative of the corresponding action functional one obtains

$$\begin{aligned} D^2S = i \sum_{e=(x,y) \in E(\mathcal{G}_W)} & \left\{ \tilde{\zeta}(X - Y + \theta_e) - \tilde{\zeta}(X - Y - \theta_e) \right\} (dX - dY)^2 \\ & + i \sum_{e=(x,y) \in E(\mathcal{G}_W)} \left\{ \tilde{\zeta}(X + Y + \theta_e) - \tilde{\zeta}(X + Y - \theta_e) \right\} (dX + dY)^2. \end{aligned}$$

Investigating the same reality conditions as above, we get the same convexity conditions.

Thus, the variational description of this formulation of (Q4) is even simpler than the one above. On the other hand, the formula of the quad-equation itself is not as convenient.

3.7 Main theorem

We summarize here the main result proven in the previous paragraphs. In the case of (Q4), we restrict to the case of purely imaginary half-period ratios τ .

Theorem 1. *Let \mathcal{G} be a finite connected bipartite quad-graph embedded in an oriented surface and let \mathcal{G}_W denote the white subgraph corresponding to a partition of $V(\mathcal{G})$. Moreover, let θ_e be a labeling of the edges $e \in E(\mathcal{G}_W)$.*

For any generalized Q-equation (i.e., a not necessarily integrable system determined by general labels θ_e), there exist reality conditions under which the corresponding action functional is strictly convex on $\mathbb{R}^{|V(\mathcal{G}_W)|}$ or the subspace U defined in Equation (10), respectively.

More precisely, such conditions are given by $X \in \mathbb{R}$ for all vertices x and the following restriction of the labels θ_e , $e \in E(\mathcal{G}_W)$:

<i>Q-equation</i>	<i>functional</i>	<i>space</i>	<i>convexity condition</i>
(Q1) $_{\delta=0}$	(9)	U	$\theta_e \in \mathbb{R}^+$
(Q1) $_{\delta=1}$	(12)	U	$\theta_e \in \mathbb{R}^+$
(Q2)	(14)	$\mathbb{R}^{ V(\mathcal{G}_W) }$	$\theta_e \in \mathbb{R}^+$
(Q3) $_{\delta=0}$	(17)	U	$\exp(i\theta_e) \in S_+^1$
(Q3) $_{\delta=1}$	(22)	$\mathbb{R}^{ V(\mathcal{G}_W) }$	$\exp(i\theta_e) \in S_+^1$
(Q4)	(27)	$\mathbb{R}^{ V(\mathcal{G}_W) }$	$\exp(i\theta_e) \in S_+^1$

Note that the restriction to the subspace U is possible if and only if (10), (13) or (18) is satisfied, respectively.

If \mathcal{G}_W is bipartite, we can partition $V(\mathcal{G}_W)$ into V_1 and V_2 . Then $X \in \mathbb{R}$, $Y \in \mathbb{R} + \pi i$ and $\exp(i\theta_e) \in S_+^1$ for all edges $e = (x, y)$ such that $x \in V_1, y \in V_2$ gives reality conditions, under which the action functional of $(Q3)_{\delta=0}$, $(Q3)_{\delta=1}$ and $(Q4)$ is strictly convex.

Under the reality conditions for the variables as above, it is not possible that $\theta_e \theta_{e'} < 0$ for two edges $e, e' \in E(\mathcal{G}_W)$ if S is convex in the cases of (Q1), (Q2) and (Q3).

Remark. Multiplying θ_e by -1 in the theorem above yields also feasible reality conditions.

In the case of a non-real half-period ratio τ of length 1, we follow the original formulation of [1], where the long leg function Φ is given by

$$\Phi(X, Y; \theta) = \frac{\sigma(X - Y + i\theta)\sigma(X + Y + i\theta)}{\sigma(X - Y - i\theta)\sigma(X + Y - i\theta)}.$$

Here $i\theta := A - B$, where A, B are certain values fulfilling $\wp(A) = \alpha, \wp(B) = \beta$. Let ω and $\omega' := \tau\omega$ be half-periods of \wp such that $\Omega := 2\omega + 2\omega' \in \mathbb{R}^+$. For the second derivative of the corresponding action functional we get

$$\begin{aligned} D^2 S = & i \sum_{e=(x,y) \in E(\mathcal{G}_W)} \{\zeta(X - Y + \theta_e) - \zeta(X - Y - \theta_e)\} (dX - dY)^2 \\ & + i \sum_{e=(x,y) \in E(\mathcal{G}_W)} \{\zeta(X + Y + \theta_e) - \zeta(X + Y - \theta_e)\} (dX + dY)^2. \end{aligned}$$

Let us consider reality conditions $X, Y, \theta_e \in \mathbb{R}$ for all edges $e = (x, y) \in E(\mathcal{G}_W)$. Let Γ be the lattice spanned by $2\omega, 2\omega'$. By construction, $\bar{\Gamma} = \Gamma$.

Proposition 1. *In the situation above, there exists $r = r(\omega) > 0$, such that S is strictly convex if $\theta_e \in (0, r)$ for all $e \in E(\mathcal{G}_W)$ or $\theta_e \in (-r, 0)$ for all $e \in E(\mathcal{G}_W)$.*

Proof. S is strictly convex if $\text{Im}(\zeta(Z + i\theta_e) - \zeta(Z - i\theta_e))$ has the same sign for all edges e and $Z \in \mathbb{R}$. Since ζ is odd, it suffices to consider the case $\theta > 0$. Since $\zeta(u + \Omega) = \zeta(u) + 2\zeta(\omega) + 2\zeta(\omega')$, we may restrict to $Z \in [0, \Omega]$.

We want to show first, that for any $Z \in \mathbb{R}$, there exists a positive real number $r(Z)$, such that $\text{Im}(\zeta(Z + i\theta) - \zeta(Z - i\theta))$ has the same sign for all $\theta \in (0, r(Z))$. If $Z \not\equiv 0 \pmod{\Omega}$, consider an open ball U around 0, such that $(Z \pm U) \cap \Gamma = \emptyset$. Then $f(u) := (\zeta(Z + u) - \zeta(Z - u))$ is holomorphic in U , non-vanishing and takes purely imaginary values if $\text{Re } u = 0$. So any $u \in U$ such that $\text{Re } u = 0$ and $\text{Im}(\zeta(Z + u) - \zeta(Z - u)) = 0$ is a zero of f . But the zeroes of f lie discrete in U . The claim in this case follows. Now consider $Z \equiv 0 \pmod{\Omega}$. But then $f \equiv 2\zeta$ and ζ has poles exactly on the points of Γ . The statement follows.

Since f is continuous also with respect to Z , we can choose $r(Z)$ depending continuously on Z . Because $[0, \Omega]$ is compact, the minimum r of $r(Z)$, $Z \in [0, \Omega]$, is positive and has the desired property.

More generally, we obtain in the same way that for any $\theta_0 \in \mathbb{R}$ there is an open interval $J \subset \mathbb{R}$ such that θ_0 is in the closure of J and S is strictly convex if $\theta_e \in J$ for all edges e . \square

3.8 Limit cases

We want to demonstrate how to obtain the setting of (Q3) as certain limits of our consideration of (Q4). As in Section 3.6, we consider the half-periods $\omega := \pi i$ and $\omega' := -t$, $t \in \mathbb{R}$. We take the limit $t \rightarrow \infty$.

Since $\text{sn} \rightarrow \sin$, the formulation of the quad-equation (Q4) given in the appendix converges to the one of $(\text{Q3})_{\delta=1}$.

Using the asymptotic behavior of the theta-functions given in the beginning of Section 3.6, we see immediately that the long leg function Φ given in (25) converges to the long leg function Φ of $(\text{Q3})_{\delta=1}$ given in (20). Also, φ defined in (26) converges to φ defined in (21).

Equation (30) shows, that

$$\begin{aligned} i\tilde{\zeta}(X \pm Y + i\theta) - i\tilde{\zeta}(X \pm Y + i\theta) &\rightarrow \frac{\sin(\theta)}{\cosh(X \pm Y) - \cos(\theta)} \\ i\tilde{\zeta}(X \pm Y \pm t + i\theta) - i\tilde{\zeta}(X \pm Y \pm t + i\theta) &\rightarrow 0. \end{aligned}$$

Thus, the second derivative of the action functional $S_{(\text{Q4})}$, given by (28), converges to (23), the second derivative of the action functional $S_{(\text{Q3})_{\delta=1}}$.

The limit of $\Lambda_{(\text{Q4})}$ defined in (27) satisfies the Conditions (6) and (7) with φ defined in (21). But the same does $\Lambda_{(\text{Q3})_{\delta=1}}$ of $(\text{Q3})_{\delta=1}$ defined as the summand of the sum over all edges in (22). Hence, their difference has to be constant in X and Y . Evaluating at $X = Y = 0$ we obtain

$$\begin{aligned} \Lambda_{(\text{Q4})}(0, 0; i\theta) &= 0, \\ \Lambda_{(\text{Q3})_{\delta=1}}(0, 0; i\theta) &= 4 \text{Im Li}_2(\exp(i\theta)). \end{aligned}$$

Therefore,

$$S_{(\text{Q4})} \rightarrow S_{(\text{Q3})_{\delta=1}} - \sum_{e=(x,y) \in E(\mathcal{G}_W)} 4 \text{Im Li}_2(\exp(i\theta_e)).$$

Remember that we parametrized $x = \text{sn}(X)$. sn has an imaginary period K , $\text{Im } K > 0$, therefore we can look at the transformation $X \mapsto X + K$ for all vertices x . This transformation does not change the value of x . But if we now consider the limit $\text{Im } K \rightarrow \infty$, $\text{Im } X \rightarrow \infty$ for all vertices x . Thus, it does not longer make sense to consider $X + Y$, since its absolute value is infinitely large. So we restrict our attention to the differences $X - Y$. If we eliminate the terms in the action functional of (Q4) involving the sum $X + Y$, we obtain the action functional of $(\text{Q3})_{\delta=0}$ defined in (17) in the limit. Also, the long leg function of (Q4) converges to the one of $(\text{Q3})_{\delta=0}$ defined in (15), because

$$\frac{\sinh\left(\frac{Z+i\theta}{2}\right)}{\sinh\left(\frac{Z-i\theta}{2}\right)} \rightarrow \exp(-i\theta)$$

as $Z \rightarrow \infty$.

In the same way, we obtain $(\text{Q3})_{\delta=0}$ as a limit of $(\text{Q3})_{\delta=1}$ by $\text{Im } X \rightarrow \infty$ for all $x \in V(\mathcal{G}_W)$. This corresponds to see the Euclidean plane as a limit of hyperbolic planes with curvature going to 0.

4 Integrable cases

We are now interested in that situations in the integrable case, where we have always positive or always negative terms in front of $(dX \pm dY)^2$ in D^2S ($e = (x, y)$ an edge), especially strict convexity of the action functional S .

Proposition 2. *In the cases of (Q1) and (Q2), the convexity conditions given in Theorem 1 can never be achieved in the integrable case if \mathcal{G} contains at least one inner white vertex.*

Proof. In the described situation, S is strictly convex if and only if all labels θ are positive (or negative).

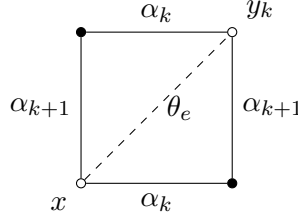


Figure 3: Induced labeling

If we consider all edges $e = (x, y_k)$ incident to an inner white vertex x , and $\theta_e = \alpha_k - \alpha_{k+1}$ is induced from a labeling of $E(\mathcal{G})$ as shown in Figure 3, then

$$\sum_{(x, y_k) \in E(\mathcal{G}_W)} \theta_e = \sum_{(x, y_k) \in E(\mathcal{G}_W)} (\alpha_k - \alpha_{k+1}) = 0. \quad (31)$$

Therefore, not all labels θ can be positive (or negative). \square

In the following, we will consider the convexity conditions of (Q3) and (Q4) given in Theorem 1.

Thus, we are exactly looking for labelings of $E(\mathcal{G})$, where the labels are real and defined up to multiples of 2π , such that for the induced labeling the convexity condition $\theta_e \equiv (0, \pi) \pmod{2\pi}$ for all $e \in E(\mathcal{G}_W)$ holds. The case of $\theta_e \equiv (\pi, 2\pi) \pmod{2\pi}$ is obtained by replacing all the labels α of $E(\mathcal{G})$ by $-\alpha$.

We define the black subgraph \mathcal{G}_B analogously to \mathcal{G}_W . Then any labeling of $E(\mathcal{G})$ induces one of $E(\mathcal{G}_B)$ in the same way as one of $E(\mathcal{G}_W)$: We simply take $\theta_{e^*} = -\theta_e$ if $e \in E(\mathcal{G}_W)$, $e^* \in E(\mathcal{G}_B)$ are diagonals of the same face $f \in F(\mathcal{G})$. This is consistent with our definition of the induced labeling, since opposite edges of any quadrilateral $f \in F(\mathcal{G})$ have the same label.

Definition. \mathcal{G} is called *simply-connected*, if the part of the surface $F(\mathcal{G})$ covers is simply-connected.

Proposition 3. *Let \mathcal{G} be simply-connected. Suppose a labeling of $E(\mathcal{G}_W)$ with real numbers is given, where the labels θ are considered the same if they differ only by a multiple of 2π . Then this labeling is integrable if and only if following two equations are satisfied for all inner white vertices x and inner black vertices x^* in \mathcal{G} :*

$$\prod_{e=(x, y_k) \in E(\mathcal{G}_W)} \exp(i\theta_e) = 1, \quad (32)$$

$$\prod_{e^*=(x^*, y_k^*) \in E(\mathcal{G}_B)} \exp(i\theta_{e^*}) = 1. \quad (33)$$

Proof. The forward implication is given by (31). For the backward implication we notice that (32) allows us to integrate in the star of any white vertex. We have to check that this defines a labeling in a consistent manner. This is the case if and only if the sum of θ_e for all white edges e of a given cycle vanishes mod 2π . Since \mathcal{G} is simply-connected, this is true if it is true for any elementary cycle (corresponding to the star of a black vertex), which is exactly the content of Equation (33). \square

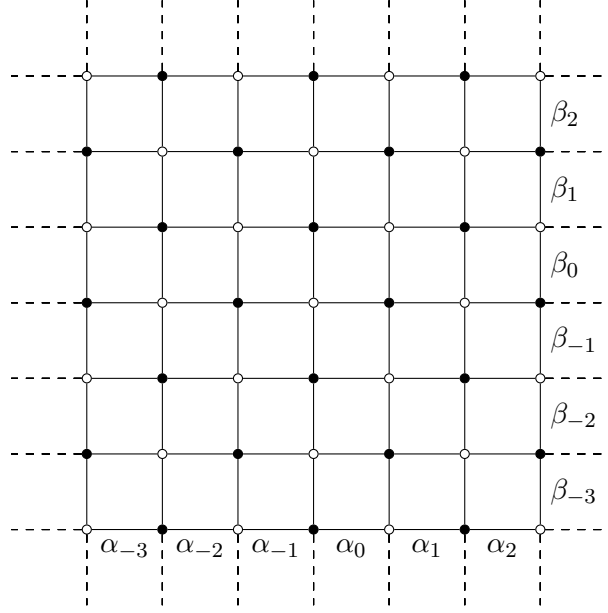


Figure 4: \mathbb{Z}^2

In what follows, we give two classes of graphs and labelings which fulfill our convexity condition.

Given a finite sub-quad-graph of \mathbb{Z}^2 , we consider only labelings of its edges that are induced from a labeling of \mathbb{Z}^2 , see Figure 4. It is easy to check that all such labelings fulfilling our convexity condition are given by the following (up to periods and adding a fixed number to all labels):

$$\begin{aligned}
&\beta_0 = 0, \\
&\alpha_{2j} \in (0, \pi) \text{ for all } j, \\
&\alpha_{2j+1} \in (\pi, 2\pi) \text{ for all } j, \\
&\beta_{2j+1} \in \left(\max_{k \in M_j} \alpha_{2k}, \min_{k \in M_j} \alpha_{2k} + \pi \right) \cap \left(\max_{k \in N_j} \alpha_{2k+1} - \pi, \min_{k \in N_j} \alpha_{2k+1} \right) \text{ for all } j, \\
&\beta_{2j} \in \left\{ \left(\max_{k \in M'_j} \alpha_{2k} - \pi, 2\pi \right) \cup \left[0, \min_{k \in M'_j} \alpha_{2k} \right] \right\} \cap \left\{ \left(\max_{k \in N'_j} \alpha_{2k+1}, 2\pi \right) \cup \left[0, \min_{k \in N'_j} \alpha_{2k+1} + \pi \right] \right\} \text{ for all } j \neq 0.
\end{aligned}$$

Here M_j, M'_j, N_j, N'_j denote the sets of all k such that there exists a quadrilateral in the graph with labels $\alpha_{2k}, \beta_{2j+1}$; α_{2k}, β_{2j} ; $\alpha_{2k+1}, \beta_{2j+1}$; $\alpha_{2k+1}, \beta_{2j}$, respectively.

Note that if α_j and β_0 are chosen as above, one can always take $\beta_{2j} = 0$ and $\beta_{2j+1} = \pi$ for all j .

It is not hard to obtain a similar result for infinite sub-quad-graphs of \mathbb{Z}^2 .

Definition. A *spider-graph* is the (infinite) quad-graph which is constructed in the following way: Take infinitely many concentric regular $2n$ -gons, $n \geq 2$, which are equally spaced. Add the radial edges between two successive polygons and divide the central polygon by adding $n - 2$ parallel diagonals into quadrilaterals.

An example of a spider-graph is given in Figure 5 ($n = 4$).

All labelings fulfilling our convexity condition are given by the following (up to periods and adding a

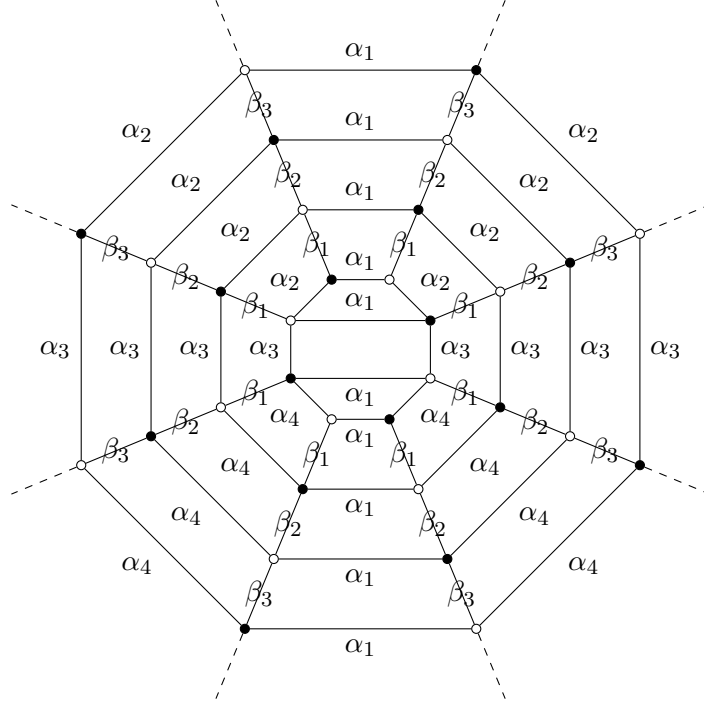


Figure 5: Spider-graph with central octagon

fixed number to all labels):

$$\begin{aligned}
\alpha_1 &= 0, \\
\alpha_{2j+1} &\in (0, \pi) \text{ for all } j > 0, \\
\alpha_{2j} &\in (\pi, 2\pi) \text{ for all } j > 0, \\
\beta_{2j+1} &\in (\max\{\max_k \alpha_{2k} - \pi, \max_k \alpha_{2k+1}\}, \pi) \text{ for all } j \geq 0, \\
\beta_{2j} &\in (\max\{\max_k \alpha_{2k} + \pi, \max_k \alpha_{2k+1}\}, \pi) \text{ for all } j > 0.
\end{aligned}$$

Definition. A *strip* in \mathcal{G} is a path of faces of the graph, such that two successive faces share an edge and the strip leaves a face in the opposite edge where it enters it. Moreover, we assume that strips have maximal length, i.e., there are no strips containing it apart from itself.

The following proposition is due to [10]:

Proposition 4. A planar quad-graph \mathcal{G} admits an embedding in \mathbb{C} with all rhombic faces if and only if the following two conditions are satisfied:

1. No strip crosses itself or is periodic.
2. Two distinct strips cross each other at most once.

Therefore, spider-graphs possess no rhombic embedding. Rhombic-embeddable graphs are an interesting and important subclass of quad-graphs.

Proposition 5. If the white subgraph \mathcal{G}_W or the black subgraph \mathcal{G}_B is bipartite, then any rhombic embedding of \mathcal{G} in \mathbb{C} (which can be ramified) yields a labeling of edges of \mathcal{G} , such that the induced

labeling on $E(\mathcal{G}_W)$ fulfills the convexity condition of (Q3) and (Q4) given in Theorem 1. If \mathcal{G}_W is bipartite, the induced labels are given by the angles θ of the rhombi at the white vertices; if \mathcal{G}_B is bipartite, the induced labels are given by the angles $\theta^* = \pi - \theta$ of the rhombi at the black vertices.

Proof. Since \mathcal{G}_W (or \mathcal{G}_B) is bipartite, we can choose a partition into two types of black (white) vertices, say of type 1 and of type 2. We orient all edges of \mathcal{G}_W , such that they always start in a white (black) and end in a black (white) vertex. See Figure 6.

To each oriented edge $\vec{e} = (x, y)$ we associate the complex number $\gamma(\vec{e}) := y - x / \|y - x\|$, where we now consider the vertices as points in \mathbb{C} . If the white (black) starting point of e is of type 1, $\gamma(\vec{e}) := \exp(i\alpha)$, otherwise $\gamma(\vec{e}) := -\exp(i\alpha)$. The labels α (which are unique up to multiples of 2π) are then defined also for non-oriented edges and yield a labeling of $E(\mathcal{G}_W)$.

The induced labels are given by $\alpha - \beta$. But $\alpha - \beta \equiv \theta \pmod{2\pi}$ ($\alpha - \beta \equiv \theta^* \pmod{2\pi}$). \square

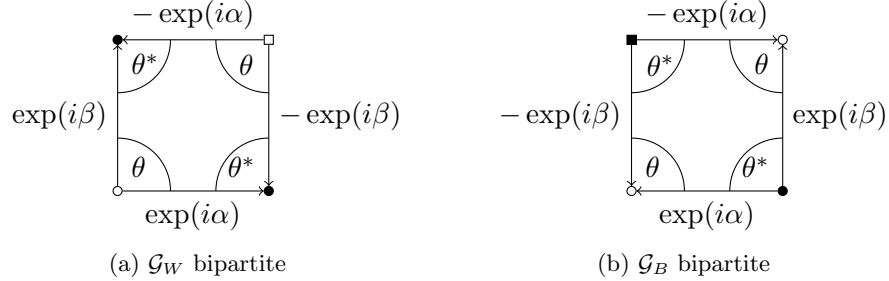


Figure 6: Induced labeling of rhombic embeddings

Note that for the case that \mathcal{G} is simply-connected, that combinatorially, \mathcal{G}_W (or \mathcal{G}_B) is bipartite if and only if all interior black (white) vertices in \mathcal{G} are of even degree.

Proposition 6. *Suppose \mathcal{G} corresponds to a cell decomposition of the disk and the white subgraph \mathcal{G}_W or the black subgraph \mathcal{G}_B is bipartite. Then any integrable labeling θ of $E(\mathcal{G}_W)$, $\theta_e \in (0, \pi)$ for all edges e , which is characterized in Proposition 3, gives a rhombic realization of \mathcal{G} (ramified embedding).*

Proof. Given an integrable labeling, we choose labels α on the edges of \mathcal{G} such that the given labeling on $E(\mathcal{G}_W)$ is induced. Now we define $\gamma(\vec{e})$ through the labels α as in the proof of Proposition 5. Choosing one starting point and the length of all edges, $\gamma(\vec{e})$ yields a rhombic realization of \mathcal{G} . \square

Proposition 7. *If \mathcal{G} contains an inner white (or black) vertex of degree 2, then any induced labeling of $E(\mathcal{G}_W)$ does not satisfy the convexity condition of (Q3) and (Q4) given in Theorem 1 (implying that the terms in front of $(dX \pm dY)^2$ in D^2S , $e = (x, y)$ an edge, all have the same sign).*

Proof. In the case of an inner white vertex we have the situation shown in Figure 7. No labeling of $E(\mathcal{G})$ induces a labeling satisfying our convexity condition due to $\theta_e = \alpha - \beta = -\theta_{e'}$. \square

A short look to Figure 7 shows, that the pair of strips is intersecting twice. Thus, any quad-graph \mathcal{G} containing an inner vertex of degree 2 possesses no rhombic embedding. But also some rhombic-embeddable cannot carry (Q3)- or (Q4)-systems fulfilling the convexity condition of Theorem 1:

Proposition 8. *There exist infinitely many rhombic-embeddable quad-graphs, such that every induced labeling does not satisfy the convexity condition for (Q3) and (Q4) given in Theorem 1.*

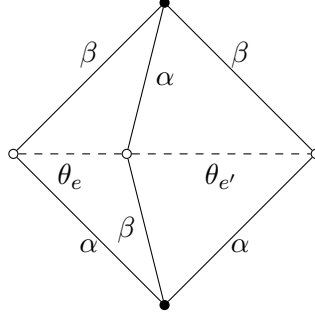


Figure 7: Inner white vertex of degree 2

Proof. It is easy to see that the graph shown in Figure 8 possesses a rhombic embedding according to Proposition 4. Let us show that every induced labeling does not satisfy our convexity condition.

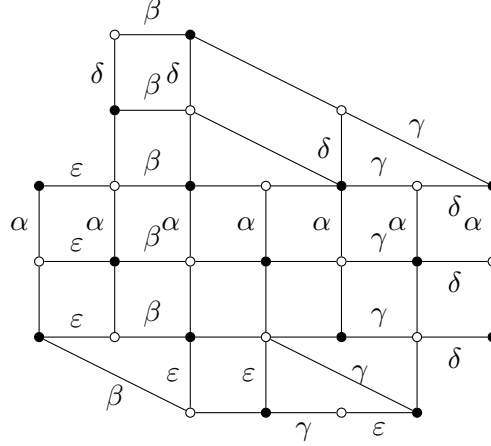


Figure 8: rhombic-embeddable graph from Proposition 8

Assume the contrary and consider an appropriate labeling of $E(\mathcal{G})$. Without loss of generality, all labels are in $[0, 2\pi)$. Let $\varepsilon = 0$. By our convexity condition, $\exp(i(\varepsilon - \mu)) \in S_+^1$ for all $\mu \in \{\alpha, \beta, \gamma\}$. Thus, $\alpha, \beta, \gamma \in (\pi, 2\pi)$. The convexity condition implies $\exp(i(\alpha - \beta)), \exp(i(\gamma - \alpha)) \in S_+^1$, which gives $\gamma > \alpha > \beta$. But also, $\exp(i(\alpha - \delta)), \exp(i(\delta - \beta)), \exp(i(\delta - \gamma)) \in S_+^1$. Therefore,

$$\begin{aligned} \delta &\in (\alpha - \pi, \alpha) \cap \{(\beta, 2\pi) \cup [0, \beta - \pi)\} \cap \{(\gamma, 2\pi) \cup [0, \gamma - \pi)\} \\ &= (\beta, \alpha) \cap \{(\gamma, 2\pi) \cup [0, \gamma - \pi)\} \\ &= (\beta, \alpha) \cap [0, \gamma - \pi) \\ &= \emptyset, \end{aligned}$$

contradiction. □

Note that the graph shown in Figure 8 fulfills the following condition: Any three pairwise intersecting strips are not all intersected by the same strip.

5 Existence and uniqueness of solutions of (Q3)- and (Q4)-Dirichlet boundary value problems

Theorem 2. *Let θ be a labeling of $E(\mathcal{G}_W)$ (not necessarily induced), such that $0 < \theta_e < \pi$ for all edges e . Moreover, let $\Theta_x \in \mathbb{R}^+$ be given. Then the corresponding action functionals (17) and (22) have an extremum (which is unique because of convexity) if and only if the following two conditions are satisfied:*

1. $\sum_{e \in E(\mathcal{G}_W)} 2\theta_e^* - \sum_{x \in V(\mathcal{G}_W)} \Theta_x$ is equal to 0 in the case of $(Q3)_{\delta=0}$ and greater than 0 in the case of $(Q3)_{\delta=1}$,
2. $\sum_{e \in E'} 2\theta_e^* - \sum_{x \in V'} \Theta_x > 0$ for all nonempty $V' \subsetneq V(\mathcal{G}_W)$ and E' being the set of all edges incident to some vertex in V' .

Proof. In Sections 3.4 and 3.5 we have seen that one can identify the action functionals in the case of (Q3) with the circle pattern functional of [5] under certain conditions (e.g. $0 < \theta < \pi$ and $\Theta > 0$). The question when extrema of this functionals exist was solved in [5]. The proof uses coherent angle systems and is relatively complicated. \square

Remark. The result in Theorem 2 holds not only in the case of $0 < \theta_e < \pi$ and $\Theta_x > 0$ for all edges e and vertices x of \mathcal{G}_W , but in an analogous way also for the other cases described in Sections 3.4 and 3.5.

In the non-geometric case, the situation simplifies if all Θ vanish.

Definition. We consider the action functional S in the case of a Q-equation. If we look for an extremum of S under the constraint, that some variables of our choice have fixed values, we will call this a *Dirichlet boundary value problem*.

Theorem 3. *Let θ be a labeling of $E(\mathcal{G}_W)$ (not necessarily induced), such that $\exp(i\theta_e) \in S_+^1$ for all edges e or $\exp(i\theta_e) \in S_-^1$ for all edges e . Then the Dirichlet boundary value problem for $(Q3)_{\delta=1}$ is uniquely solvable, if $\Theta_x = 0$ for all vertices x corresponding to a non-fixed variable.*

Proof. The uniqueness follows from strict convexity of the action functional. Without loss of generality, we may assume $0 < \theta_e < \pi$ for all edges e . Then for some constant C , $S - C$ is equal to

$$\begin{aligned} & \sum_{e=(x,y) \in E(\mathcal{G}_W)} \{ \operatorname{Im} \operatorname{Li}_2(\exp(X - Y + i\theta_e)) + \operatorname{Im} \operatorname{Li}_2(\exp(Y - X + i\theta_e)) \} \\ & + \sum_{e=(x,y) \in E(\mathcal{G}_W)} \{ \operatorname{Im} \operatorname{Li}_2(\exp(X + Y + i\theta_e)) + \operatorname{Im} \operatorname{Li}_2(\exp(-X - Y + i\theta_e)) \}. \end{aligned}$$

Now for any $Z \in \mathbb{R}$ and $0 < \theta < \pi$, we have

$$\operatorname{Im} \operatorname{Li}_2(\exp(Z + i\theta)) + \operatorname{Im} \operatorname{Li}_2(\exp(-Z + i\theta)) > (\pi - \theta) |Z|.$$

Hence, we obtain

$$\begin{aligned}
S &> \sum_{e=(x,y) \in E(\mathcal{G}_W)} (\pi - \theta_e)(|X - Y| + |X + Y|) + C \\
&\geq \min_{e \in E(\mathcal{G}_W)} (\pi - \theta_e) \sum_{e=(x,y) \in E(\mathcal{G}_W)} (|X - Y| + |X + Y|) + C \\
&= \min_{e \in E(\mathcal{G}_W)} (\pi - \theta_e) \sum_{e=(x,y) \in E(\mathcal{G}_W)} 2 \min(|X|, |Y|) + C.
\end{aligned}$$

So we cannot find a minimum if some variable X is not bounded. Our claim immediately follows. \square

Note that according to Equation (5), we cannot consider the condition $\Theta_x = 0$ for some vertices x in the case of $(Q3)_{\delta=0}$, since φ is either always positive or always negative in the case that $\exp(i\theta_e) \in S_+^1$ for all edges e or $\exp(i\theta_e) \in S_-^1$ for all edges e , respectively.

Theorem 4. *Let θ be a labeling of $E(\mathcal{G}_W)$ (not necessarily induced), such that the convexity conditions given in Theorem 1 and the subsequent remark are fulfilled. Then the Dirichlet boundary value problem for (Q4) is uniquely solvable.*

Proof. The uniqueness follows from convexity.

$$\begin{aligned}
\frac{\partial \varphi}{\partial X}(X, Y; \theta) &= i(\tilde{\zeta}(X + Y + i\theta) - \tilde{\zeta}(X + Y - i\theta)) + i(\tilde{\zeta}(X + Y + t + i\theta) - \tilde{\zeta}(X + Y + t - i\theta)) \\
&\quad + i(\tilde{\zeta}(X - Y + i\theta) - \tilde{\zeta}(X - Y - i\theta)) + i(\tilde{\zeta}(X - Y - t + i\theta) - \tilde{\zeta}(X - Y - t - i\theta))
\end{aligned}$$

is $2t$ -periodic in X and Y and smooth on \mathbb{R}^2 if $\theta \not\equiv 0 \pmod{2\pi}$. Thus, it is bounded. The same is true for $\partial \varphi / \partial Y$. Therefore, $\varphi(\cdot, \cdot, \theta)$ is Lipschitz-continuous. Since the set of edges is finite, we may choose the Lipschitz constant L uniformly for all θ_e .

Elementary calculations yield:

$$\Phi(X - \pi\tau, Y; \theta) = \Phi(X, Y; \theta) \exp(8i\theta), \quad (34)$$

$$\Phi(X, Y - \pi\tau; \theta) = \Phi(X, Y; \theta). \quad (35)$$

Now choose $\mu = \mu(X, Y; \theta), \eta = \eta(X, Y; \theta)$ such that

$$\begin{aligned}
\varphi(X + 2t, Y; \theta) - \varphi(X, Y; \theta) &= \mu(X, Y; \theta), \\
\varphi(X, Y + 2t; \theta) - \varphi(X, Y; \theta) &= \eta(X, Y; \theta).
\end{aligned}$$

For continuity reasons, η is constant and $\mu = \mu(\theta)$ depends continuously only on $\theta \in (0, 2\pi)$. Since $\Phi(X, Y; \pi) \equiv 1$, $\eta = 0$. Because of Equation (30) and the subsequent remark, $\partial \varphi / \partial X > 0$, so $\mu > 0$.

Without loss of generality, $\exp(i\theta_e) \in S_+^1$ for all edges e . Assume that the Dirichlet boundary value problem is not solvable. Then there exists a sequence $(\{X_j(n)\}_{j=1}^v)_{n=0}^\infty$, $v = |V(\mathcal{G})|$, of variables (some of them fixed), such that at least one variable is unbounded and the sequence $(S(n))_{n=0}^\infty$, $S(n) := S(X_1(n), \dots, X_v(n))$, is strictly decreasing.

Without loss of generality, the X_1 -variable is unbounded. By passing to a subsequence, which we will now and in the future denote exactly as the original one, we can achieve $X_1(n) \rightarrow \pm\infty$ strictly

monotonously. In the same way, we obtain for any j that $X_j(n)$ either converges to a constant a_j (being fixed is a special case), or strictly monotonously to $\pm\infty$.

Let $X_j(n) \rightarrow \pm\infty$. If we consider this sequence on the compact circle $\mathbb{R}/2t\mathbb{Z}$, we can choose a convergent subsequence. I.e., there is $a_j \in \mathbb{R}$ and a sequence $(m_j(n))_{n=0}^\infty$ of integers such that

$$X_j(n) - 2m_j(n)t \rightarrow a_j.$$

We may assume that $m_j(n) \rightarrow \pm\infty$ strictly monotonously. With the same notation, we can include the case that $X_j(n)$ converges to a constant by having $m_j(n) = 0$ for all n .

Let $0 < \varepsilon < 2t$. By passing to a subsequence, we can achieve

$$|X_j(n) - 2m_j(n)t - a_j| < \varepsilon$$

for all j and n . By our considerations above, we have for any edge $e = (x_j, x_k)$:

$$\begin{aligned} \varphi(X_j(n), X_k(n'); \theta_e) &= \varphi(a_j + 2m_j(n)t, a_k + 2m_k(n')t; \theta_e) + B(X_j(n), X_k(n'); \theta_e) \\ &= \varphi(a_j, a_k; \theta_e) + \mu(\theta_e)m_j(n) + B(X_j(n), X_k(n'); \theta_e), \end{aligned}$$

where $|B(X_j(n), X_k(n'); \theta_e)| < L\sqrt{2\varepsilon^2}$. More general, it holds for any real $X = a + 2pt + q$, $p \in \mathbb{Z}$ and $|q| < 2t$, that

$$\varphi(X, X_k(n'); \theta_e) = \varphi(a, a_k; \theta_e) + \mu(\theta_e)p + B(X, X_k(n'); \theta_e),$$

where $|B(X, X_k(n'); \theta_e)| < L\sqrt{\varepsilon^2 + 4t^2}$.

For $M > 0$, define

$$p_{\max} := (L\sqrt{\varepsilon^2 + 4t^2} + M + \max_{X_j(n) \text{ diverges}} \max_{e=(x_j, x_k)} |\varphi(a_j, a_k; \theta_e)|) \frac{1}{\min_{e \in E(\mathcal{G}_W)} \mu(\theta_e)}.$$

Then for $X = a + 2pt + q$ as above and $|p| > p_{\max}$,

$$\varphi(X, X_k(n'); \theta_e) \geq \pm M$$

if $p \geq 0$, provided that $e = (x_j, x_k)$, $X_j(n)$ diverges. By passing to a subsequence, we may assume $|m_j(n)| > p_{\max}$ for all j such that $X_j(n)$ diverges.

Let

$$C := \sqrt{2}L\varepsilon + \max_{X_j(n) \text{ converges}} \max_{e=(x_j, x_k)} |\varphi(a_j, a_k; \theta_e)|.$$

Then $|\varphi(X, X_k(n'); \theta_e)| < C$ whenever $e = (x_j, x_k)$, $X_j(n)$ converges and $|X - a_j| < \varepsilon$.

Now let

$$E := \max_{x \in V(\mathcal{G}_W)} |\Theta_x|, \tilde{C} := C + E, \tilde{M} := M - E.$$

Choose M such that $\tilde{M} > 0$.

We have shown that

$$\begin{aligned} \frac{\partial S}{\partial X_j}(X_1(n+1), \dots, X_{j-1}(n+1), X, X_{j+1}(n), \dots, X_v(n)) &= \sum_{x_k: e=(x_j, x_k) \text{ edge}} \varphi(X_j, X_k(n \text{ or } n+1); \theta_e) \\ &\quad + \Theta_{x_j} \\ &\geq \pm \tilde{M}, \end{aligned}$$

if $X \in \text{conv}\{X_j(n), X_j(n+1)\}$ and $X_j(n) \rightarrow \pm\infty$; and

$$\begin{aligned} & \left| \frac{\partial S}{\partial X_j}(X_1(n+1), \dots, X_{j-1}(n+1), X, X_{j+1}(n), \dots, X_v(n)) \right| \\ &= \left| \sum_{x_k: e=(x_j, x_k) \text{ edge}} \varphi(X_j, X_k(n \text{ or } n+1); \theta_e) + \Theta_{x_j} \right| \\ &< \tilde{C}v, \end{aligned}$$

if $X \in \text{conv}\{X_j(n), X_j(n+1)\}$ and $X_j(n) \rightarrow a_j$.

Without loss of generality, $X_j(n) \rightarrow \pm\infty$ if and only if $j \leq k$. If we integrate along the piecewise straight path

$$\begin{aligned} & (X_1(n), \dots, X_v(n)) - (X_1(n+1), X_2(n), \dots, X_v(n)) \\ & \quad - (X_1(n+1), X_2(n+1), \dots, X_v(n)) \\ & \quad - \dots \\ & \quad - (X_1(n+1), \dots, X_v(n+1)), \end{aligned}$$

we obtain

$$\begin{aligned} S_{n+1} - S_n &> \tilde{M} \sum_{j=1}^k |X_j(n+1) - X_j(n)| - \tilde{C}v \sum_{j=k+1}^v |X_j(n+1) - X_j(n)| \\ &> \tilde{M} |X_1(n+1) - X_1(n)| - \tilde{C}v(v-k)2\varepsilon. \end{aligned}$$

By passing to a subsequence, we can achieve $|X_1(n+1) - X_1(n)| \geq 1$ for all n . If we choose M such that $M > E + 2\tilde{C}v(v-k)\varepsilon$, then $S_{n+1} - S_n > 0$, contradiction. Therefore, the Dirichlet boundary value problem is solvable. \square

Remark. This theorem is also true in the case of rhombic lattices (half-period ratio τ has length 1) as mentioned in the end of Section 3.7. For this, we use that

$$\sigma(u + \Omega) = -\exp(2\eta'(u + \omega + \omega'))\sigma(u),$$

where ω and $\omega' := \tau\omega$ are half-periods of \wp such that $\Omega := 2\omega + 2\omega' \in \mathbb{R}^+$, $\eta' = \zeta(\omega) + \zeta(\omega')$. We obtain then results analogous to (34) and (35), such that the same argumentation works.

6 Integrable circle patterns via (Q3)

In this section, we want to describe a connection between (Q3) and circle patterns as described in [5]. Figure 9 shows two circles, centered at x and y with radii r_x and r_y , which intersect at the angle θ_e (the circles being canonically oriented). The fact that the angles around an inner black vertex sum up to 2π corresponds to Equation (33) in Proposition 3.

An elementary calculation yields

$$\exp(2i\phi_e^x) = \frac{r_x - r_y \exp(-i\theta_e)}{r_x - r_y \exp(i\theta_e)} \quad (36)$$

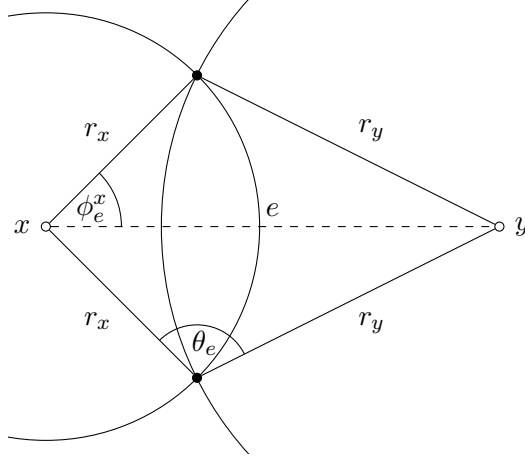


Figure 9: Intersection of circles

in the Euclidean case and

$$\exp(2i\phi_e^x) = \log \left(\frac{\tanh(\frac{r_x}{2}) - \tanh(\frac{r_y}{2}) \exp(-i\theta_e)}{\tanh(\frac{r_x}{2}) - \tanh(\frac{r_y}{2}) \exp(i\theta_e)} \right) - \log \left(\frac{1 - \tanh(\frac{r_x}{2}) \tanh(\frac{r_y}{2}) \exp(-i\theta_e)}{1 - \tanh(\frac{r_x}{2}) \tanh(\frac{r_y}{2}) \exp(i\theta_e)} \right) \quad (37)$$

in the hyperbolic case.

Let

$$r_x = \exp(X), r_y = \exp(Y) \text{ in the Euclidean case,} \quad (38)$$

$$\tanh\left(\frac{r_x}{2}\right) = \exp(X), \tanh\left(\frac{r_y}{2}\right) = \exp(Y) \text{ in the hyperbolic case,} \quad (39)$$

$X, Y \in \mathbb{R}$. This is possible because all radii are positive. It is easy to check, that in both cases

$$2\phi_e^x = -\varphi(X, Y; \theta_e),$$

where φ is defined in (16) for $(Q3)_{\delta=0}$ or in (21) for $(Q3)_{\delta=1}$ corresponding to the Euclidean and hyperbolic case, respectively.

Thus, if we choose Θ_x to be the cone angle at the vertex $x \in V(\mathcal{G}_W)$, we see the correspondence between (Q3) and circle patterns with intersection angles θ under the reality conditions $X \in \mathbb{R}$, $\theta_e \in (0, \pi)$ for all vertices x and edges e .

By Proposition 3, the corresponding labeling of $E(\mathcal{G}_W)$ is integrable if and only if

$$\prod_{e=(x,y_k) \in E(\mathcal{G}_W)} \exp(i\theta_e) = 1 \quad (40)$$

is true for all inner white vertices x .

Assume now the circle pattern can be decomposed into two sets of circles, such that any two circles from one set have no points in common. Equivalently, \mathcal{G}_W is bipartite (so $V(\mathcal{G}_W) = V_1 \dot{\cup} V_2$). Then there is a way to connect (Q3) with such circle patterns, due to the second reality conditions of (Q3) which differ to the previous ones by adding πi to X if $x \in V_2$. If we replace X by $X + \pi i$ (or Y by $Y + \pi i$) in (38) or (39), we have to replace r_x by $-r_x$ (or r_y by $-r_y$) and therefore θ_e by $-\theta_e^*$ such

that $\exp(2i\phi_e^x)$ in Equations (36) and (37) remains the same. According to the end of Sections 3.4 and 3.5, we obtain another correspondence between (Q3), now with labels $-\theta^*$ instead of θ , and circle patterns with intersection angles θ .

The corresponding labeling of $E(\mathcal{G}_W)$ is then integrable if and only if

$$\prod_{e=(x,y_k) \in E(\mathcal{G}_W)} \exp(i\theta_e^*) = 1 \quad (41)$$

is true for all inner white vertices x .

Equation (40) is equivalent to (41) if and only if all vertices of \mathcal{G}_W have even degree. If \mathcal{G}_W is simply-connected, this corresponds to the black subgraph \mathcal{G}_B being bipartite.

If \mathcal{G} corresponds to a cell decomposition of a disk, circle patterns fulfilling Equation (41) are called *integrable* in [7]. Only then the circle patterns admit an isoradial realization. Observe the correspondence to Propositions 5 and 6.

Note that our definition of integrability, i.e., the corresponding labeling of $E(\mathcal{G}_W)$ is induced, is in [7] denoted as *integrability of the corresponding cross-ratio system*.

Acknowledgment

We would like to thank V.V. Bazhanov and Yu. B. Suris for numerous fruitful discussions.

Appendix: ABS list

In the following, we will give the equations in the list Q of the ABS list. We also show how the formulation we use is related to the one of [8] by transformations of the variables or parameters.

For labels α, β , let $\theta := \alpha - \beta$. Generalized Q-equations considered in Section 3 correspond to a general choice θ .

List Q:

$$\begin{aligned} (\text{Q1})_{\delta=0}: \quad Q &= \alpha(xu + yv) - \beta(xv + yu) - (\alpha - \beta)(xy + uv), \\ \varphi(x, y; \theta) &= \frac{i\theta}{x-y}; \end{aligned}$$

The transformation $\alpha \mapsto -i\alpha$ for all parameters α relates the long leg function to the one in [8] given by

$$\varphi(x, y; \theta) = \frac{\theta}{x-y}.$$

The quad equations are the same in both formulations.

$$\begin{aligned} (\text{Q1})_{\delta=1}: \quad Q &= \alpha(xu + yv) - \beta(xv + yu) - (\alpha - \beta)(xy + uv) - \alpha\beta(\alpha - \beta), \\ \Phi(x, y; \theta) &= \frac{x-y+i\theta}{x-y-i\theta}; \end{aligned}$$

The transformation $\alpha \mapsto -i\alpha$ for all parameters α relates our formulation to the one in [8], where $+\alpha\beta(\alpha - \beta)$ instead of $-\alpha\beta(\alpha - \beta)$ appears in Q and

$$\Phi(x, y; \theta) = \frac{x-y+\theta}{x-y-\theta}.$$

$$\begin{aligned}
(\text{Q2}): \quad Q &= \alpha(xu + yv) - \beta(xv + yu) - (\alpha - \beta)(xy + uv) \\
&\quad - \alpha\beta(\alpha - \beta)(x + u + y + v) - \alpha\beta(\alpha - \beta)(\alpha^2 - \alpha\beta + \beta^2), \\
x &= X^2, \\
\Phi(x, y; \theta) &= \frac{(X+Y+i\theta)(X-Y+i\theta)}{(X+Y-i\theta)(X-Y-i\theta)};
\end{aligned}$$

The transformation $\alpha \mapsto -i\alpha$ for all parameters α relates our formulation to the one in [8], where $+\alpha\beta(\alpha - \beta)(x + u + y + v)$ instead of $-\alpha\beta(\alpha - \beta)(x + u + y + v)$ appears in Q and

$$\Phi(x, y; \theta) = \frac{(X + Y + \theta)(X - Y + \theta)}{(X + Y - \theta)(X - Y - \theta)}.$$

$$\begin{aligned}
(\text{Q3})_{\delta=0}: \quad Q &= \sin(\alpha)(xu + yv) - \sin(\beta)(xv + yu) - \sin(\alpha - \beta)(xy + uv), \\
x &= \exp(X), \\
\Phi(x, y; \theta) &= \exp(-i\theta) \frac{\sinh\left(\frac{X-Y+i\theta}{2}\right)}{\sinh\left(\frac{X-Y-i\theta}{2}\right)};
\end{aligned}$$

The transformation $X \mapsto iX$ for all variables X relates (up to equivalence) the long leg function to the one in [8] given by

$$\Phi(x, y; \theta) = \frac{\sin\left(\frac{X-Y+\theta}{2}\right)}{\sin\left(\frac{X-Y-\theta}{2}\right)}.$$

The quad equations are the same in both formulations, but $x = \exp(iX)$ in [8].

$$\begin{aligned}
(\text{Q3})_{\delta=1}: \quad Q &= \sin(\alpha)(xu + yv) - \sin(\beta)(xv + yu) - \sin(\alpha - \beta)(xy + uv) \\
&\quad + \sin(\alpha - \beta) \sin(\alpha) \sin(\beta), \\
x &= \cosh(X), \\
\Phi(x, y; \theta) &= \frac{\sinh\left(\frac{X+Y+i\theta}{2}\right) \sinh\left(\frac{X-Y+i\theta}{2}\right)}{\sinh\left(\frac{X+Y-i\theta}{2}\right) \sinh\left(\frac{X-Y-i\theta}{2}\right)};
\end{aligned}$$

The transformation $X \mapsto i(X - \pi/2)$ for all variables X relates the long leg function to the one in [8] given by

$$\Phi(x, y; \theta) = \frac{\cos\left(\frac{X+Y+\theta}{2}\right) \sin\left(\frac{X-Y+\theta}{2}\right)}{\cos\left(\frac{X+Y-\theta}{2}\right) \sin\left(\frac{X-Y-\theta}{2}\right)}.$$

The quad equations are the same in both formulations, but $x = \sin(X)$ in [8].

$$\begin{aligned}
(\text{Q4}): \quad Q &= \text{sn}(\alpha)(xu + yv) - \text{sn}(\beta)(xv + yu) - \text{sn}(\alpha - \beta)(xy + uv) \\
&\quad + \text{sn}(\alpha - \beta) \text{sn}(\alpha) \text{sn}(\beta)(1 + \kappa^2 xuyv), \\
x &= \text{sn}(-iX + \pi/2), \\
\Phi(x, y; \theta) &= \frac{\vartheta_1\left(\frac{X+Y+i\theta}{2i}\right) \vartheta_4\left(\frac{X+Y+i\theta}{2i}\right) \vartheta_1\left(\frac{X-Y+i\theta}{2i}\right) \vartheta_4\left(\frac{X-Y+i\theta}{2i}\right)}{\vartheta_1\left(\frac{X+Y-i\theta}{2i}\right) \vartheta_4\left(\frac{X+Y-i\theta}{2i}\right) \vartheta_1\left(\frac{X-Y-i\theta}{2i}\right) \vartheta_4\left(\frac{X-Y-i\theta}{2i}\right)}.
\end{aligned}$$

The transformation $X \mapsto i(X - \pi/2)$ for all variables X relates the long leg function to the one in [8] given by

$$\Phi(x, y; \theta) = \frac{\vartheta_2\left(\frac{X+Y+\theta}{2}\right) \vartheta_3\left(\frac{X+Y+\theta}{2}\right) \vartheta_1\left(\frac{X-Y+\theta}{2}\right) \vartheta_4\left(\frac{X-Y+\theta}{2}\right)}{\vartheta_2\left(\frac{X+Y-\theta}{2}\right) \vartheta_3\left(\frac{X+Y-\theta}{2}\right) \vartheta_1\left(\frac{X-Y-\theta}{2}\right) \vartheta_4\left(\frac{X-Y-\theta}{2}\right)}.$$

The quad equations are the same in both formulations, but $x = \text{sn}(X)$ in [8].

References

- [1] V.E. Adler, A.I. Bobenko, and Yu.B. Suris. Classification of integrable equations on quad-graphs. The consistency approach. *Commun. Math. Phys.*, 233(3):513–543, 2003.
- [2] V.V. Bazhanov, V.V. Mangazeev, and S.M. Sergeev. Faddeev-Volkov solution of the Yang-Baxter equation and discrete conformal symmetry. *Nucl. Phys. B*, 784(3):234–258, 2007.
- [3] V.V. Bazhanov and S.M. Sergeev. A master solution of the quantum Yang-Baxter equation and classical discrete integrable equations. arXiv:1006.0651, 2010.
- [4] V.V. Bazhanov and S.M. Sergeev. Elliptic gamma-function and multi-spin solutions of the Yang-Baxter equation. arXiv:1106.5874, 2011.
- [5] A.I. Bobenko and B.A. Springborn. Variational principles for circle patterns and Koebe’s theorem. *Trans. Amer. Math. Soc.*, 356(2):659–689, 2004.
- [6] A.I. Bobenko and Yu.B. Suris. Integrable systems on quad-graphs. *Intern. Math. Research Notices*, 2002(11):573–611, 2002.
- [7] A.I. Bobenko and Yu.B. Suris. *Discrete differential geometry: integrable structures*, volume 98 of *Grad. Stud. in Math.* AMS, Providence, 2008.
- [8] A.I. Bobenko and Yu.B. Suris. On the Lagrangian structure of integrable quad-equations. *Lett. Math. Phys.*, 92(3):17–31, 2010.
- [9] A. Hurwitz. *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen*. Springer-Verlag, Berlin, 2000. 5. Auflage.
- [10] R. Kenyon and J.-M. Schlenker. Rhombic embeddings of planar quad-graphs. *Trans. Amer. Math. Soc.*, 357(9):3443–3458, 2005.
- [11] S. Lobb and F.W. Nijhoff. Lagrangian multiforms and multidimensional consistency. *J. Phys. A: Math. Theor.*, 42(45), 2009. 454013 (18pp).
- [12] J.E. Marsden, G.W. Patrick, and S. Shkoller. Multisymplectic geometry, variational integrators and nonlinear PDEs. *Commun. Math. Phys.*, 199(2):351–395, 1998.
- [13] J. Moser and A.P. Veselov. Discrete versions of some classical integrable systems and factorization of matrix polynomials. *Commun. Math. Phys.*, 139(2):217–243, 1991.
- [14] F.W. Nijhoff. Lax pair for the Adler (lattice Krichever-Novikov) system. *Phys. Lett. A*, 297(1-2):49–58, 2002.
- [15] Yu.B. Suris. Discrete Lagrangian models. In B. Grammaticos, Y. Kosmann-Schwarzbach, and T. Tamizhmani, editors, *Lecture Notes Phys.*, volume 644, pages 111–184. Springer, Berlin, 2004.